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Role shape operátoru v kalibračních teoriích

The Role of Shape Operator in Gauge Theories

MASTER'S THESIS

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V Praze dne ....

podpis

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Šimon Vedl

Název práce:
Role shape operátoru v kalibračních teoriích

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Abstrakt: Kalibrační teorie jsou velmi prominentní napříč fyzikou. Nejjednodušším příkladem kalibrační teorie je elektromagnetismus. V této práci se inspirujeme přístupem k popisu vnořených variet za pomoci shape operátoru and pokoušíme se jej zobecnit pro hlavní fibrované prostory. Výchozím bodem je věta o univerzální konexi od Narasimhana a Ramanana. Ukážeme, že univerzální konexe může být využita k odstranění kalibrační volnosti. Navíc dokážeme zlepšit dimenzionální nároky na konstrukci univerzální konexe. Odhalíme analog shape operátoru a rotujícího podprostoru. A nakonec přeformulujeme Yang-Millsovu a Palatiniho akci v řeči rotujícího podprostoru a prozkoumáme, jak se změní příslušné pohybové rovnice.
Klíčová slova: shape operátor, univerzální konexe, kalibrační teorie

Title:
The Role of Shape Operator in Gauge Theories

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Abstract: Gauge theories are omnipresent in physics, with the simplest example being the electromagnetic theory. In this thesis, we take inspiration from the treatment of embedded manifolds using the shape operator and attempt to generalize it for principal bundles. The starting point is the theorem by Narasimhan and Ramanan about universal connections. We show that the universal connection can be used to cancel gauge freedom. Furthermore, we also slightly improve dimensional requirements of the construction of universal connection. We discover an analog of shape operator and rotating blade. Finally, we rephrase Yang-Mills and Palatini action in terms of the rotating blade and investigate how the equations of motion change for these theories.
Keywords: shape operator, universal connection, gauge theory

Contents

Introduction

Symmetry is a very crucial concept in physics. From the homogeneity of cosmic fluid through the structure of crystals down to the fundamental interactions in particle physics, symmetry permeates reality around us. We use its presence to simplify equations and its absence to detect exciting phenomena like phase transitions. Symmetries usually correspond to some transformation that does not change a property we care about. For example, the square looks the same even if we rotate it by ninety degrees about its center. Symmetries have several important properties. They can be reversed and composed to give another symmetry. Mathematicians call this a group structure.

In this thesis, we are going to focus on gauge symmetries. These are often mathematical redundancies present in theory by default. Examples may include the choice of a locally free-falling coordinate system in general relativity, the phase factor of a wave function in quantum physics, or the orientation of a field in its internal space. This gauge freedom is often advantageous for specific calculations. However, one runs into problems with overcounting when describing all possible configurations of a system. For example, this is particularly bothersome for the path integral formulation of quantum field theory.

The mathematical tool used to describe gauge theories is called the principal bundle. These spaces resemble a Cartesian product of spacetime and a group when examined very closely. The goal of this thesis is inspired by the description of Riemannian manifolds embedded in a Euclidean space. One can view the spherical shell as either an abstract smooth manifold or as the actual surface in the usual three-dimensional space. To quote David Hestenes[1]: "The treatment of intrinsic geometry can be simplified by coordinating it with extrinsic geometry." Based on this view, one might ask the following questions. Is there an analog of embedding in the case of principal bundles? Where do we embed the principal bundle? How is the mapping realized? Furthermore, how does the shape operator fit in? We hope to answer these in this thesis.

In the first chapter, we revisit the description of embedded Riemannian mani-

[1]D. Hestenes: *The Shape of Differential Geometry in Geometric Calculus*, In: L. Dorst and J. Lasenby (eds.): *Guide to Geometric Algebra in Practice*, Springer (2011)

folds. We mainly focus on introducing covariant derivative on the tangent vector fields and extending it to any vector field on the manifold. In doing so, we encounter the shape operator and the rotating blade. Both are geometrically interpreted using Clifford algebra at the end of the chapter.

The second chapter introduces concepts from the theory of fiber bundles. We define what constitutes a bundle and what is a connection on a fiber bundle. Then we focus on two particular kinds of bundles. The first is a principal bundle. As mentioned above, this is the natural home of gauge freedom. Next, we discuss vector bundles which are usually used as spaces where matter fields live. Finally, we discuss the relationship between vector bundles and principal bundles.

Symmetries are the central theme of the third chapter. First, we recall basic definitions and properties concerning Lie groups and corresponding Lie algebras. Then we introduce homogeneous spaces and explore their rich geometry. The main result will be the classification of invariant connections on a homogeneous space. Lastly, we are going to mention the theorem by Narasimhan and Ramanan [19] about universal connections.

Inspired by the proof of the theorem about universal connections, we set out to investigate geometry surrounding the Grassmannian. The Grassmannian is a set of subspaces of a given dimension in a certain vector space. Chapter four probes the various bundles over the Grassmannian and the corresponding canonical connections. Then we introduce a gauge-invariant description of connection and discover the shape operator and rotating blade analogous to those in embedded geometry. After that, we discuss the pullback of this canonical structure to other principal bundles using the concept of universal connection and discuss possible ambiguities of this approach. In the second part of the chapter, we interpret the rotating blade as the dynamical variable and formulate its equations of motion.

Finally, the fifth chapter focuses on particular gauge theories. First, we recast electromagnetic theory using the rotating blade and shape operator. We visit the geometrically interesting Dirac monopole and show that the rotating blade is defined globally even though the potentials are singular. Then we give an improved method for finding the universal connection for an arbitrary electromagnetic field. Next, we discuss a method for finding the universal connection for non-abelian Yang-Mills theories. The final part of this chapter focuses on gravity. There we introduce the Palatini formalism and formulate equations of motion for the rotating blade in the context of gravity.

Chapter 1

Embedded Riemannian manifolds

This chapter is going to focus on Riemannian manifolds that can be embedded into a Euclidean space. In fact we consider a relaxed setting with a smooth manifold M^d of dimension d and a smooth map $f : M^d \to \mathbb{E}^N$, where N is sufficiently large and the differential of f has maximal rank everywhere, i.e. f is an immersion [10, p.169]. If M^d is already equipped with a metric g, we demand f to be an isometric immersion, i.e., it holds

$$g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) \equiv g_{\mu\nu} = \frac{\partial f}{\partial x^\mu} \cdot \frac{\partial f}{\partial x^\nu}, \tag{1.1}$$

where the dot is the Euclidean inner product and x^μ, $\mu = 1, \dots, d$, are coordinates on M^d. The bounds on N are given by Cartan-Janet and Nash theorems. If one is interested only in local immersion the minimal N given by the Cartan-Janet theorem [4, p.98] is $N = d(d+1)/2$. The bound for a global immersion is given by theorem of Nash [15, p.354] as $N = n(3n+11)/2$.

From now on we use the shorthand $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and $f_\mu \equiv \partial_\mu f$. If M^d has no metric, we can induce it by the equation (1.1). Thanks to the map f, one can also work with column vectors in $\mathbb{E}^N$ instead of abstract vector fields on the manifold M^d. This is because of the one-to-one correspondence $\partial_\mu \leftrightarrow \partial_\mu f$. For computational convenience, one also introduces an auxiliary basis of the tangent space f^μ called the *reciprocal* basis satisfying the relation

$$f^\mu \cdot f_\nu = \delta^\mu_\nu. \tag{1.2}$$

The metric g grants the manifold M^d the unique Levi-Civita connection on the tangent bundle. It allows us to transport tangent vectors along smooth curves. Given a smooth vector field X and a curve $x^\mu(s)$ the equation for the parallel transport reads

$$0 = \frac{dX^\mu}{ds} + \Gamma^\mu{}_{\nu\sigma} X^\sigma \frac{dx^\nu}{ds} = \frac{dx^\nu}{ds}\left(\frac{\partial X^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\nu\sigma} X^\sigma\right), \tag{1.3}$$

where

$$\Gamma^{\mu}{}_{\nu\sigma} = \frac{1}{2} g^{\mu\rho}(\partial_\sigma g_{\rho\nu} + \partial_\nu g_{\sigma\rho} - \partial_\rho g_{\nu\sigma}) \tag{1.4}$$

are the Christoffel symbols of the second kind. We denote the term in brackets in eq. (1.3) as $(\nabla_\nu X)^\mu$ and call it the covariant derivative of the vector field X in the direction of coordinate x^ν.

Since the map f induces the metric, the Levi-Civita connection is also induced by f. However, the connection can also be obtained in a more elegant, straightforward way by projecting the usual partial derivative to the tangent plane

$$(\nabla_\nu X)^\mu f_\mu = f_\mu \, f^\mu \cdot \partial_\nu (X^\sigma f_\sigma) = f_\mu \left((f^\mu \cdot f_\sigma) \partial_\nu X^\sigma + f^\mu \cdot (\partial_\nu f_\sigma) X^\sigma \right). \tag{1.5}$$

It can easily be shown that $\Gamma^{\mu}{}_{\nu\sigma} = f^\mu \cdot (\partial_\nu f_\sigma)$ by substituting for metric from (1.1) and using the symmetry of partial derivatives $\partial_\mu f_\nu = \partial_\nu f_\mu$.

At this point, we could attach to every point of the manifold a copy of the Euclidean space $\mathbb{E}^N$ in a similar way one attaches to each point the tangent space. In other words, one considers a bundle over M^d with typical fiber $\mathbb{E}^N$. The map f provides a *solder form* which maps vector fields (sections of the tangent bundle) to $\mathbb{E}^N$-valued functions (sections of the $\mathbb{E}^N$ bundle over M^d). It essentially shows that the $\mathbb{E}^N$ bundle can be viewed as the direct sum of vector bundles [2, p.210] of the tangent and normal bundle. For a generic section v we can extend the covariant derivative (1.5) to

$$D_\mu v = \mathsf{P}_\parallel \partial_\mu (\mathsf{P}_\parallel v) + \mathsf{P}_\perp \partial_\mu (\mathsf{P}_\perp v) = \partial_\mu v + (\mathsf{P}_\parallel \partial_\mu \mathsf{P}_\parallel + \mathsf{P}_\perp \partial_\mu \mathsf{P}_\perp) v, \tag{1.6}$$

where $\mathsf{P}_\parallel$ is the projection onto the space spanned by $f_1, \ldots, f_d$ and $P_\perp$ the projection onto the orthogonal complement. We define the shape operator to be an operator-valued one-form S with components

$$\mathsf{S}_\mu = \mathsf{P}_\parallel \partial_\mu \mathsf{P}_\parallel + \mathsf{P}_\perp \partial_\mu \mathsf{P}_\perp. \tag{1.7}$$

To explore the matrix structure of S_μ, we fix a basis n_α in the normal space, where $\alpha = d+1, \ldots, N$, and use its reciprocal basis to express the projector as $\mathsf{P}_\perp v = n_\alpha \, n^\alpha \cdot v$. First, we examine the action of S_μ on the tangent vectors f_ν

$$\mathsf{S}_\mu \, f_\nu = \mathsf{P}_\parallel (\partial_\mu \mathsf{P}_\parallel) f_\nu + \mathsf{P}_\perp (\partial_\mu \mathsf{P}_\perp) f_\nu \tag{1.8}$$
$$= \mathsf{P}_\parallel \partial_\mu (\mathsf{P}_\parallel f_\nu) - \mathsf{P}_\parallel \mathsf{P}_\parallel \partial_\mu f_\nu + \mathsf{P}_\perp \partial_\mu (\mathsf{P}_\perp f_\nu) - \mathsf{P}_\perp \mathsf{P}_\perp \partial_\mu f_\nu \tag{1.9}$$
$$= -P_\perp \partial_\mu f_\nu = -n_\alpha \, n^\alpha \cdot (\partial_\mu f_\nu). \tag{1.10}$$

For normal vectors n_α one obtains similarly

$$\mathsf{S}_\mu \, n_\alpha = -f_\nu \, f^\nu \cdot (\partial_\mu n_\alpha) = f^\nu \, n_\alpha \cdot (\partial_\mu f_\nu). \tag{1.11}$$

Therefore, in the basis given by f_μ and n_α the following block structure can be inferred

$$\mathsf{S}_\mu = \begin{pmatrix} 0 & -\mathsf{s}_\mu^T \\ \mathsf{s}_\mu & 0 \end{pmatrix}, \tag{1.12}$$

where s_μ is a matrix with $N-d$ rows and d columns with components

$$(\mathsf{s}_\mu)_{\alpha\nu} = (\partial_\mu n_\alpha) \cdot f_\nu, \tag{1.13}$$

which in the case of two dimensional surfaces in three dimensional Euclidean space reproduces (up to a choice of sign) the components of the second fundamental form [5, p.143].

There is an alternative prescription for the shape operator that is computationally more advantageous. By realizing $\mathsf{P}_\perp = 1 - \mathsf{P}_\parallel$, one gets

$$\mathsf{S}_\mu = \mathsf{P}_\parallel \partial_\mu \mathsf{P}_\parallel + \mathsf{P}_\perp \partial_\mu \mathsf{P}_\perp = \frac{1}{2}(2\mathsf{P}_\parallel - 1)\partial_\mu(2\mathsf{P}_\parallel - 1) = \frac{1}{2}\mathsf{R}\partial_\mu \mathsf{R}, \tag{1.14}$$

where $\mathsf{R} = 2\mathsf{P}_\parallel - 1$ is the reflection about the tangent space which we call *rotating blade*. This is closely related with the generalized Gauss map [23, p.277], which assigns to every point the tangent space as a subspace of $\mathbb{E}^N$. Using this form, it is easy to calculate the following expression, where one only uses the fact that $\mathsf{R}^2 = 1$ and therefore $(\partial_\mu \mathsf{R})\mathsf{R} = -\mathsf{R}\,\partial_\mu \mathsf{R}$

$$\partial_\mu \mathsf{S}_\nu - \partial_\nu \mathsf{S}_\mu = (\partial_\mu \mathsf{R})\mathsf{R}^2(\partial_\nu \mathsf{R}) - (\partial_\nu \mathsf{R})\mathsf{R}^2(\partial_\mu \mathsf{R}) = -2[\mathsf{S}_\mu, \mathsf{S}_\nu]. \tag{1.15}$$

The curvature $\Omega_{\mu\nu}$ associated with the covariant derivative D_μ is defined via the usual commutator of covariant derivatives

$$\Omega_{\mu\nu} v = [D_\mu, D_\nu] v = (\partial_\mu \mathsf{S}_\nu - \partial_\nu \mathsf{S}_\mu + [\mathsf{S}_\mu, \mathsf{S}_\nu]) v. \tag{1.16}$$

Using the equation (1.15), the curvature simplifies to

$$\Omega_{\mu\nu} = -[\mathsf{S}_\mu, \mathsf{S}_\nu] = \frac{1}{4}[\partial_\mu \mathsf{R}, \partial_\nu \mathsf{R}] = [\partial_\mu \mathsf{P}_\parallel, \partial_\nu \mathsf{P}_\parallel]. \tag{1.17}$$

Due to the off-diagonal structure of S, the matrix $\Omega_{\mu\nu}$ is block-diagonal. The Riemann curvature tensor is recovered from the block corresponding to the tangent space

$$R^\rho{}_{\sigma\mu\nu} = f^\rho \cdot (\Omega_{\mu\nu} f_\sigma). \tag{1.18}$$

The other block represents the curvature on the normal space.

1.1 The multivector perspective

The geometric interpretation of the shape operator is much clearer if one introduces Clifford algebra on the Euclidean space $\mathbb{E}^N$ [25]. The quickest way to construct Clifford algebra is to take the exterior algebra $\bigwedge \mathbb{E}^N$, which is a graded algebra under the exterior product, and define the Clifford product between a vector $v \in \mathbb{E}^N$ and an r-vector $A \in \bigwedge^r \mathbb{E}^N$ as

$$vA = v \cdot A + v \wedge A, \tag{1.19}$$

where $v \cdot A$ is the usual interior product. The Clifford product is associative, distributive over addition and for vectors it satisfies

$$v^2 = |v|^2. \tag{1.20}$$

The exterior and interior product can be rewritten entirely in terms of the Clifford product

$$v \cdot A = \frac{1}{2}\left(vA - (-1)^r Av\right), \tag{1.21}$$

$$v \wedge A = \frac{1}{2}\left(vA + (-1)^r Av\right). \tag{1.22}$$

Reimagining vectors f_μ as part of the Clifford algebra we can form the following object

$$f_1 \wedge f_2 \wedge \ldots \wedge f_d = \sqrt{\det g}\, I_M, \tag{1.23}$$

where I_M is called the pseudoscalar of the manifold M. It is also an example of a blade. Blades are object that can be written as an exterior product of vectors. There is a one-to-one correspondence between blades and subspaces [12, p.17] and can be used to define projectors. This way we get the projector P onto the tangent space

$$\mathsf{P}(v) = (v \cdot I_M) I_M^{-1}, \tag{1.24}$$

where $I_M^{-1} = (-1)^{d(d-1)/2} I_M$. The covariant derivative defined in (1.6) can be rewritten as

$$\begin{aligned} D_\mu v &= ((\partial_\mu v_\parallel) \cdot I_M) I_M^{-1} + ((\partial_\mu v_\perp) \wedge I_M) I_M^{-1} \qquad (1.25)\\ &= \partial_\mu v - (v_\parallel \cdot \partial_\mu I_M) I_M^{-1} - (v_\parallel \cdot I_M) \partial_\mu I_M^{-1} - (v_\perp \wedge \partial_\mu I_M) I_M^{-1} - (v_\perp \wedge I_M) \partial_\mu I_M^{-1}. \end{aligned}$$

Using the identities (1.21), (1.22) and the relations $v_\parallel I_M = (-1)^{d-1} I_M v_\parallel$ and $v_\perp I_M = (-1)^d I_M v_\perp$ one finally recovers

$$D_\mu v = \partial_\mu v - \frac{1}{2}(v\, I_M^{-1} \partial_\mu I_M - I_M^{-1}(\partial_\mu I_M) v) = \partial_\mu v - v \cdot (I_M^{-1} \partial_\mu I_M). \tag{1.26}$$

We denote the object in the brackets as $S_\mu = I_M^{-1} \partial_\mu I_M$ and call it shape tensor [12, p.154],[7, p.208]. The shape tensor is easy to interpret geometrically. For every μ, the S_μ is a bivector in $\mathbb{E}^N$ that infinitesimally rotates the pseudoscalar I_M so that it remains tangent to the manifold. The difference between shape operator S_μ and the shape tensor S_μ is only superficial as there is a one-to-one correspondence between skew-symmetric maps (matrices) and bivectors [12, p.80].

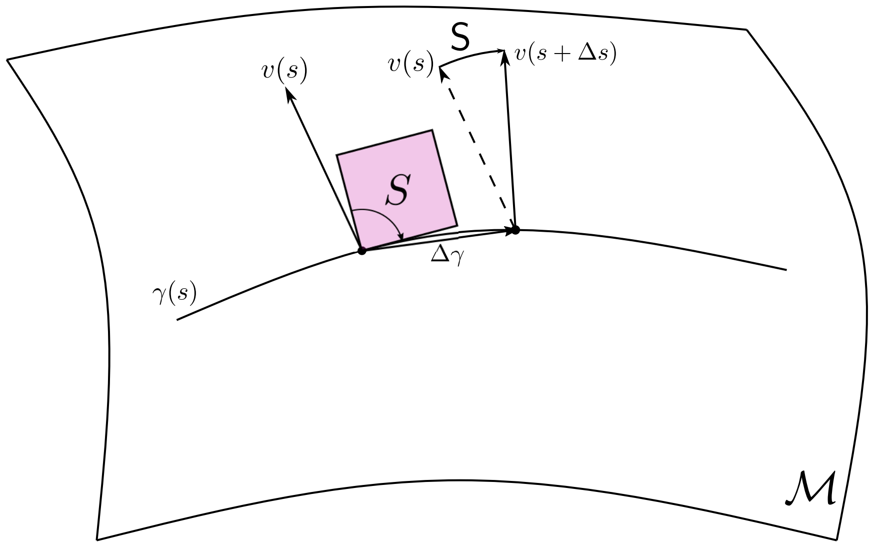

Figure 1.1: The difference between the shape tensor and shape operator. Shape tensor $S = \dot{\gamma}^\mu S_\mu$ is the pink bivector and shape operator $\mathsf{S} = \dot{\gamma}^\mu \mathsf{S}_\mu$ is the infinitesimal rotation generated by that bivector.

Chapter 2

Connections on fiber bundles

Fiber bundles are the natural language of gauge theories. In this chapter, we are going to lay out the basic definitions and terminology. To define a fiber bundle, one needs three ingredients, two smooth manifolds E and M and a smooth surjective map $\pi : E \to M$. Then one asserts some conditions on these inputs to produce a structure in which the manifold E can be locally viewed as a Cartesian product $M \times F$, where F is a smooth manifold called the *standard fiber*. F is the model manifold for all of the fibers $E_m = \pi^{-1}(\{m\})$ meaning every fiber is diffeomorphic to F. The following definition summarizes this in a more concrete form [9, p.482]

Definition 2.1. Let $\pi : E \to M$ be a smooth map (*projection*) of a smooth manifold E (*total space*) onto smooth manifold M (*base space*). Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of M and $\{\varphi_\alpha\}_{\alpha\in I}$ be a collection of smooth diffeomorphisms $\varphi_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)$, where F is a smooth manifold (*standard fiber*), such that

$$(\pi \circ \varphi_\alpha)(m, f) = m, \quad m \in U_\alpha,\ f \in F. \tag{2.1}$$

We call such combination $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ a local trivialization. We call the triplet (E, M, π) a *fiber bundle* if there exists a local trivialization.

Sometimes one denotes fiber bundles as "$\pi : E \to M$" or simply E. Simplest example of a fiber bundle is the *trivial bundle* $E = M \times F$.

The naive idea is to assume that fields are functions $\psi : M \to F$. Every function f corresponds to a *section* $\sigma_f : M \to M \times F$ of the trivial bundle via $\sigma_f(m) = (m, f(m))$. Fields, however, are not always this simple. Take, for example, a vector field on a manifold. It assigns to every point a vector in the tangent space at that point. That is the issue. The field no longer has values in a single vector space. Rather, it takes values in a vector space attached to that point. Alternatively, we can say that the vector field maps from the manifold M to

$$TM = \bigcup_{m \in M} T_m M, \tag{2.2}$$

which is called the tangent bundle. That is the reason for introducing the notion of a section

Definition 2.2. Let $\pi : E \to M$ be a fiber bundle and $\sigma : M \to E$ a smooth function. If $\pi \circ \sigma$ is the identity map then we call σ a *section* of the fiber bundle. The set of sections is denoted by $\Gamma(E)$.

At this point, one would like to talk about isomorphisms between fiber bundles. However, one has to first define a morphism between fiber bundles

Definition 2.3. Let $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$ be fiber bundles. A morphism F between these fiber bundles consists of two smooth maps $F : E_1 \to E_2$ and $\bar{F} : M_1 \to M_2$ such that the following diagram is commutative

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\bar{F}} & M_2 \end{array}$$

A morphism F is called isomorphism if both F and $\bar{F}$ are diffeomorphisms.

The fact that sections take values in different spaces over each point makes differentiation of sections difficult because differentiation involves essentially subtracting values of very close points, and there is no canonical way of mapping one fiber to another. To resolve this issue, one introduces the concept of a connection.

Definition 2.4. Let $\pi : E \to M$ be a fiber bundle. One defines the *vertical bundle* VE as a subbundle of the tangent bundle TE by

$$VE = \bigcup_{p \in E} V_p E, \qquad V_p E = \ker \pi_*|_p, \tag{2.3}$$

where π_* denotes the tangent map (differential) of π. A *connection* on a fiber bundle is a subbundle HE of the tangent bundle TE which is complementary to the vertical bundle VE.

The subbundle HE is sometimes called the horizontal bundle because it corresponds to the tangent bundle of the base manifold M. On the other hand, the name vertical bundle is supposed to evoke a correspondence with the tangent bundle of the standard fiber.

An equivalent definition of connection involves a *connection one-form* A. That is a differential one-form on TE with values in VE satisfying

1. $A \circ A = A$,

2. $VE = A(TE)$.

Essentially, A at each point $p \in E$ plays the role of the projector to the vertical space along $H_pE = \ker A|_p$. The horizontal bundle may not always be involutive which means that the commutator of two horizontal fields may have a non-trivial vertical component. That component is what we call curvature [16, p.78].

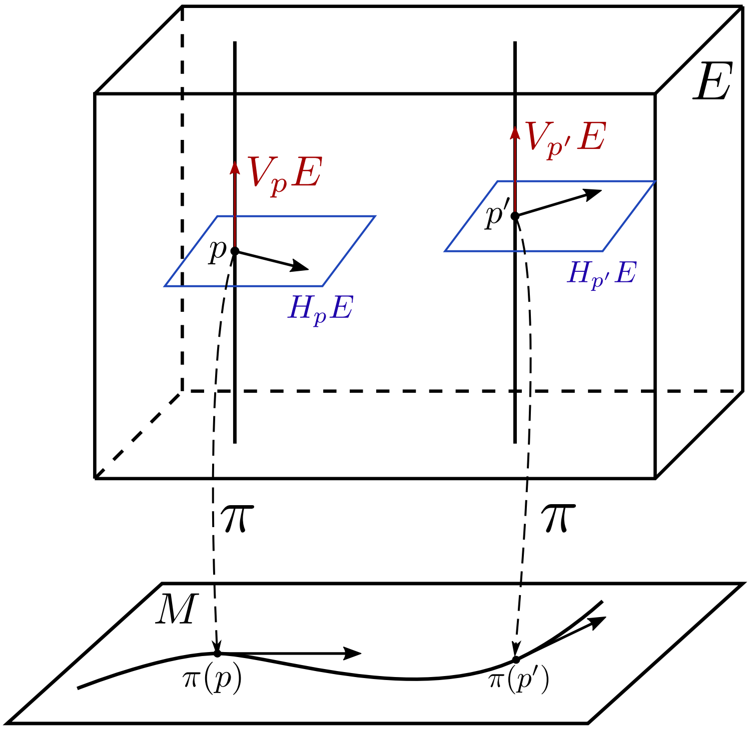

Figure 2.1: Illustration of horizontal and vertical subspace together with lifts of vector fields from the base manifold.

Once the fiber bundle is equipped with a connection, one can uniquely lift tangent vectors v on M to tangent vectors v^h on E by demanding that

$$\pi_*(v^h) = v. \tag{2.4}$$

It also allows us to lift curves from M to E by demanding the tangent vector of the lift to be horizontal. The horizontal lift of curves is sometimes referred to as *parallel transport*. At this point, we leave the setting of generic fiber bundles and discuss two types of fiber bundles relevant to gauge theories.

2.1 Principal bundles

We start with the principal bundle. One could consider a fiber bundle with a standard fiber being a Lie group H. The definition of a fiber bundle only assumes diffeomorphisms between fibers and thus completely ignores the group structure of H. However, one can salvage a relict of the group structure by assuming that the group H can act on the fibers by a free and transitive right action (c.f. section 3.1). That emulates the right group multiplication, which has the same properties. That gives the principal bundle [9, p.552]

Definition 2.5. Let $\pi : P \to M$ be a fiber bundle with a standard fiber being a Lie group H and let $R : P \times H \to P$ be an action of the group H. The fiber bundle is called a *principal H-bundle* if the following is satisfied

1. R acts along fibers: $\pi \circ R(\bullet, h) = \pi$ for any $h \in H$,
2. R is free: If $R(p, h) = p$ then $h = e$,
3. R is transitive: for any $p, p' \in P$ such that $\pi(p) = \pi(p')$ there exists $h \in H$ such that $R(p, h) = p'$,
4. There exists a local trivialization $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ such that φ_α are equivariant

$$\varphi_\alpha(m, h_1 h_2) = R(\varphi_\alpha(m, h_1), h_2). \tag{2.5}$$

In the rest of this thesis, we are going to adopt the following notation for the right action $R(p, h) \equiv R_h(p) \equiv p \cdot h$. Using the right action, one can map the Lie algebra $\mathfrak{h}$ of H to the tangent space T_pP at the point $p \in P$ via the exponential map

$$\#_p(X) = \frac{d}{dt}\bigg|_{t=0} p \cdot \exp(tX), \quad X \in \mathfrak{h}. \tag{2.6}$$

One can show that the map $\#_p$ is a linear isomorphisms of $\mathfrak{h}$ and V_pP[9, p.560]. We call $\#(X)$ *fundamental vector field* or infinitesimal generator of the right action. The connection one-form is supposed to be a projector onto the vertical bundle. Due to the isomorphism, it is sufficient to consider the connection one-form A having values in the Lie algebra $\mathfrak{h}$. That is sometimes referred to as the global connection form. The curvature of a principal connection can be calculated using the Cartan structure equations

$$F = dA + \frac{1}{2}[A \wedge A], \tag{2.7}$$

where $[\alpha \wedge \beta]$ is defined as the combination of the exterior product on forms and the commutator product on the Lie algebra $\mathfrak{h}$

$$[\alpha \wedge \beta] = \alpha^a \wedge \beta^b \, [E_a, E_b], \tag{2.8}$$

where E_a is some basis of the Lie algebra [9, p.225].

The sections of a principal bundle $\pi : P \to M$ correspond to a choice of gauge. To see this, let $\sigma, \sigma' \in \Gamma(P)$ be sections and denote pullbacks

$$\mathcal{A} = \sigma^*(A), \quad \mathcal{A}' = \sigma'^*(A), \quad \sigma'(m) = \sigma(m) \cdot \mathsf{h}(m), \tag{2.9}$$

where $\mathsf{h} : M \to H$ is called *gauge transformation*, and $\mathcal{A}, \mathcal{A}'$ are called local connection forms or *gauge potentials*. Using the following property of the connection one-form

$$R_h^*(A) = \mathrm{Ad}_{h^{-1}}(A), \tag{2.10}$$

where Ad is the adjoint representation (for definition see equation (3.12)), one can calculate

$$\mathcal{A}' = \mathrm{Ad}_{\mathsf{h}^{-1}}(\mathcal{A}) + \mathsf{h}^*(\omega_{MC}), \tag{2.11}$$

where ω_{MC} is the left Maurer-Cartan form on H. For a matrix Lie group, one recovers the usual transformation of a gauge potential

$$\mathcal{A}' = \mathsf{h}^{-1}\mathcal{A}\mathsf{h} + \mathsf{h}^{-1}d\mathsf{h}. \tag{2.12}$$

2.2 Vector bundles

The other important type of bundle is the vector bundle [2, p.205]. This term encapsulates for instance the tensor bundles such as the tangent and cotangent bundle of a manifold. The vector bundle $\pi : E \to M$ is a fiber bundle with a standard fiber being a n-dimensional vector space V. We further assume there exists a local trivialization $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ such that

$$\varphi_\alpha(m, v_1 + v_2) = \varphi_\alpha(m, v_1) + \varphi_\alpha(m, v_2). \tag{2.13}$$

Because the tangent space of a vector space is isomorphic to the vector space, we get that at each point, the vertical space is isomorphic to the fiber the point lies in. Due to the vector space structure, one can introduce the notion of a basis of sections. Sections $e_1, e_2, \ldots, e_n$ form a local basis of sections [2, p.208] if for any section $s \in \Gamma(E)$ there exist (locally) unique functions $s^1, s^2, \ldots, s^n \in C^\infty(M)$ such that

$$s = s^i e_i. \tag{2.14}$$

Essentially, sections $e_1, e_2, \ldots, e_n$ form the basis of the fiber at each point. Only a trivial bundle admits a global basis of sections [2, p.208].

Let us denote A the connection one-form and identify fibers with vertical spaces. Then for a section $s \in \Gamma(E)$, we define the covariant derivative of s in the direction of ∂_μ as

$$D_\mu(s) = A(s_*(\partial_\mu)) = A((\partial_\mu s^i)\, e_i + s^j e_{j*}(\partial_\mu)) = (\partial_\mu s^i + A_{\mu\ j}^{\ i} s^j) e_i, \tag{2.15}$$

where $e_1, e_2, \ldots, e_n$ is a local basis of sections and $A_{\mu\ j}^{\ i} = (A(e_{j*}(\partial_\mu)))^i$ are the matrix elements of the vertical part of the pushforward of ∂_μ under the map e_j.

In the setting of vector bundles, the gauge group H arises from the so-called transition functions [2, p.212]. Given a local trivialization $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$, for $U_\alpha \cap U_\beta \neq \emptyset$ we define transition functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \to H$ as

$$(\varphi_\alpha^{-1} \circ \varphi_\beta)(m, v) = \rho(h_{\alpha\beta}(m))v, \tag{2.16}$$

where ρ is a representation of the group H on the vector space V.

One can actually represent sections of a vector bundle $p : E \to M$ with gauge group H on a principal H-bundle $\pi : P \to M$ using the functions of type ρ. In general given a representation ρ on a vector space V a differential form of type ρ is a differential form α with values in the vector space V such that

$$R_h^*(\alpha) = \rho(h^{-1})\alpha. \tag{2.17}$$

Then one can introduce the operator of exterior covariant derivative D by taking the horizontal part of the usual exterior derivative. Taking the horizontal part means that the inputs are first projected to the horizontal space and then inserted into the form. Given a connection form A, the exterior covariant derivative of a function ψ of type ρ reads

$$D\psi = d\psi + d\rho(A)\psi, \tag{2.18}$$

where $d\rho$ is the derived representation (c.f. section 3.1 paragraph concerning representations).

The original vector bundle is reconstructed by constructing the so-called associated vector bundle. First one considers the Cartesian product $P \times V$ equipped with the right action of the group H

$$\mathfrak{R}((p, v), h) = (p \cdot h, \rho(h^{-1})v). \tag{2.19}$$

Then one constructs the space of orbits $(P \times V)/H$ under this action which turns out to be a vector bundle denoted $P \times_\rho V$ over M and its transition functions are $\rho(h_{\alpha\beta})$, where $h_{\alpha\beta}$ were transition functions on $\pi : P \to M$ [9, p.606]. One can show that there is a one-to-one correspondence between sections of $P \times_\rho V \to M$ and so-called H-invariant sections of $P \times V \to P$, which satisfy

$$s(p \cdot h) = \mathfrak{R}_h(s), \quad s \in \Gamma(P \times V). \tag{2.20}$$

The correspondence is very simple it is the orbit corresponding to the value of s. The section $s \in \Gamma(P \times V)$ can be written as

$$s(p) = (p, \psi(p)), \tag{2.21}$$

where $\psi : P \to V$ is a function of type ρ when s is an H-invariant section.

Chapter 3

Homogeneous spaces and universal connections

3.1 Lie groups

In the previous chapter, we encountered several topics from the theory of Lie groups and Lie algebras. In this section, we are going to consolidate these different notions. That is especially useful because the upcoming section about homogeneous spaces is going to rely heavily on Lie groups.

We first start with a notion of a group. *Group* is a set G equipped with two operations called multiplication and inverse and a distinct element e called the unity. These together must satisfy three axioms

1. Multiplication is associative: $a(bc) = (ab)c$
2. Unity: $ea = ea = a$
3. Inverse: $a^{-1}a = aa^{-1} = e$

Groups were initially invented to model symmetries of objects. We are going to be interested in *Lie groups*. Those are groups that have a smooth manifold structure that is compatible with the group operations in a way that multiplication and inverse are smooth maps. An example of a Lie group is the general linear group $GL(V)$ on a vector space V. It is the group of invertible matrices.

There are two kinds of special maps on a Lie group G. They are the left and right translations. *Left translation* L of an element $m \in G$ by an element $g \in G$ is

$$L(g, m) \equiv L_g(m) = gm. \tag{3.1}$$

And similarly for *right translation* R

$$R_g(m) = mg. \tag{3.2}$$

For a given $g \in G$, both translations are smooth maps and, in fact, diffeomorphisms because the inverse is the translation given by the inverse element g^{-1}. Diffeomorphisms have regular differentials. That can be used to show that the tangent bundle of a Lie group is trivial [1, p.13]. We will not go into much detail, but this can be done by constructing a global basis of sections obtained by fixing a basis of the tangent space at the unity and mapping it to other tangent spaces via the differential of left (or right) translation.

This leads to the concept of left-invariant vector fields. Values of such vector field at two different points are related via the left translation. Equivalently vector field X is *left-invariant* if

$$\forall g \in G \quad L_{g*}(X) = X. \tag{3.3}$$

One can show that the commutator of two left-invariant vector fields is again left-invariant. Therefore we call the set of left-invariant vector fields $\mathfrak{X}_L$ *Lie algebra of the Lie group* G, where the Lie bracket is the commutator of vector fields. These vector fields are uniquely defined by value at a single point. From this we obtain the following isomorphism $\mathfrak{g} \equiv T_e G \simeq \mathfrak{X}_L$. Conceptually speaking, this allows us to define a product between vectors. We call it the commutator product and the Lie bracket of vector fields induces it

$$[X, Y] = [X^L, Y^L]|_e, \tag{3.4}$$

where $X^L, Y^L \in \mathfrak{X}_L$ denote the vector fields corresponding to $X, Y \in \mathfrak{g}$. For the group of invertible matrices, the Lie algebra is the space of all matrices, and the Lie bracket is the matrix commutator.

Of particular interest are the integral curves of left-invariant vector fields. These are in one-to-one correspondence with one-parameter subgroups of G [1, p.16]. One can then define the *exponential map* $\exp : \mathfrak{g} \to G$ by

$$\exp(tX) = \gamma_X(t), \quad t \in \mathbb{R}, \tag{3.5}$$

where $\gamma_X : \mathbb{R} \to G$ is a homomorphism such that $\gamma_X'(0) = X$. For matrix Lie groups the exponential reduces to the usual matrix exponential

$$\exp(tX) = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n. \tag{3.6}$$

The relationship between left-invariant fields and elements of $T_e G \equiv \mathfrak{g}$ is sometimes described using the *left Maurer-Cartan form*. This is a differential one-form ω_{MC} with values in $\mathfrak{g}$ which acts on left-invariant vector field X as

$$\omega_{MC}(X) = X|_e. \tag{3.7}$$

This defines it completely as any vector field can be written as a combination of left-invariant vector fields. The Maurer-Cartan form satisfies the following [9, p.225]

$$d\omega_{MC} + \frac{1}{2}[\omega_{MC} \wedge \omega_{MC}] = 0, \tag{3.8}$$

$$L_g^*(\omega_{MC}) = \omega_{MC}, \quad R_g^*(\omega_{MC}) = \mathrm{Ad}_{g^{-1}}(\omega_{MC}), \tag{3.9}$$

where $[\alpha \wedge \beta]$ is the combination of exterior and commutator product

$$[\alpha \wedge \beta] = \alpha^a \wedge \beta^b \, [E_a, E_b], \tag{3.10}$$

where E_a is some basis of $\mathfrak{g}$ and Ad_g is the adjoint representation of G on $\mathfrak{g}$ defined as

$$\mathrm{Ad}_g(X) = \left.\frac{d}{dt}\right|_{t=0} \left(g \exp(tX) g^{-1}\right). \tag{3.11}$$

The Maurer-Cartan form can be interpreted as the flat connection on the trivial principal bundle $\pi : G \times G \to G$.

If groups are meant to represent symmetries then actions of groups correspond to transformations of an object given by that symmetry. Let M be a set then *left action* of a Lie group G on the set M is a map $\lambda : G \times M \to M$ such that

1. $\lambda(e, m) = m$ for all $m \in M$,
2. $\lambda(g_1, \lambda(g_2, m)) = \lambda(g_1 g_2, m)$ for all $g_1, g_2 \in G$ and $m \in M$.

One often adopts the following notation $\lambda(g, m) \equiv \lambda_g(m) \equiv g \cdot m$. One can also consider *right actions* which satisfy $\theta(g_1, \theta(g_2, m)) = \theta(g_2 g_1, m)$. Every left action λ can be made into right action by setting $\theta_g = \lambda_{g^{-1}}$. All actions we encounter in this text are going to possess either or both of the following properties. We say action is

- *transitive* if every pair of points $m, m' \in M$ can be related via some $g \in G$: $m' = g \cdot m$,
- *free* if $g \cdot m = m$ implies $g = e$.

Given a left action on M we can define for $m \in M$ the following sets

- *stabilizer* of m (stabilizer group): $G_m = \{g \in G |\, g \cdot m = m\}$,
- *orbit* of m: $G \cdot m = \{g \cdot m |\, g \in G\}$.

In the case when $M = V$ is a vector space we have a special kind of action called linear action. This means that every group element acts on V via a linear operator. Thus we obtain a group homomorphism $\rho : G \to GL(V)$ which we call *representation* of G on V. For every representation $\rho : G \to GL(V)$ there is also the *derived representation* $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ which is a homomorphisms of Lie algebras [9, p.247]. On every Lie group there exists the *adjoint representation* $\mathrm{Ad} : G \to GL(\mathfrak{g})$ given by

$$g \mapsto \mathrm{Ad}_g(X) = \frac{d}{dt}\bigg|_{t=0} \big(g \exp(tX) g^{-1}\big) = (L_g \circ R_{g^{-1}})_*(X). \tag{3.12}$$

The associated infinitesimal representation is denoted $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and acts as

$$X \mapsto \mathrm{ad}_X(Y) = [X, Y]. \tag{3.13}$$

Every Lie group can also be equipped with a metric of sorts. More specifically there exists a canonical symmetric Ad-invariant bilinear form $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined as

$$K(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y). \tag{3.14}$$

We call it the *Killing form*. It also satisfies the following relation

$$K([Z, X], Y) + K(X, [Z, Y]) = 0 \Leftrightarrow K(\mathrm{ad}_Z(X), Y) + K(X, \mathrm{ad}_Z(Y)) = 0, \tag{3.15}$$

which can be interpreted as saying that ad_Z is a skew-symmetric operator with respect to the inner product given by K. The Killing form is negative definite for semisimple Lie groups and thus can provide inner product in the usual sense.

3.2 Homogeneous spaces

In the first chapter, we encountered the rotating blade, which assigns to every point a subspace. The set of all n-dimensional subspaces of an N-dimensional vector space is called a Grassmannian $Gr(n, N)$. The Grassmannian is a smooth manifold, and the following construction obtains the smooth structure. Fix a subspace V_0. The orthogonal group $G = O(N)$ acts transitively on the set of subspaces. If a generic subspace V is obtained from V_0 in two different ways $V = g \cdot V_0 = g' \cdot V_0$ then the product $g^{-1}g'$ must belong to the stability group of V_0 which is $H \simeq O(n) \times O(N - n)$. Thus every subspace corresponds to an equivalence class of group elements, where the equivalence is given by

$$g \sim g' \Leftrightarrow g^{-1}g' \in H. \tag{3.16}$$

These equivalence classes are called left cosets and denoted gH, where g is some representative element. The set of these cosets is the factor space denoted G/H.

If G is a Lie group and H its closed subgroup, then the set G/H can be made into a smooth manifold. For the Grassmannian, we obtain

$$Gr(n,N) \simeq O(N)/(O(n) \times O(N-n)). \tag{3.17}$$

This section will explore intriguing structures surrounding these kinds of manifolds.

Definition 3.1 (Homogeneous space)**.** Let G be a Lie group and $H \subset G$ a closed subgroup. Then the coset space G/H is called a *homogeneous space.*

Equivalently, one can define homogeneous space as a smooth manifold equipped with transitive left action of a Lie group G.

Homogeneous spaces are equipped with natural left action λ of the group G that descends from the left translation on G

$$\lambda(g, mH) \equiv \lambda_g(mH) = (gm)H, \quad g \in G. \tag{3.18}$$

This action is naturally lifted to the tangent bundle $T(G/H)$ for any $g \in G$ via its differential λ_{g*}. Let us denote $\pi : G \to G/H$ the quotient map $\pi(g) = gH$ and let $o \equiv \pi(e) = H$. The quotient map satisfies relations for any $g \in G$ and $h \in H$

$$\pi \circ L_g = \lambda_g \circ \pi, \tag{3.19}$$
$$\pi \circ R_h = \pi. \tag{3.20}$$

The tangent space at o is given as the image of $\mathfrak{g}$ under the differential of π. The kernel of $\pi_*|_e$ is $\mathfrak{h}$ the Lie algebra of H [6, p.53] and we get the isomorphism

$$T_o(G/H) \cong \mathfrak{g}/\ker(\pi_*|_e) = \mathfrak{g}/\mathfrak{h}. \tag{3.21}$$

It is illuminating to inspect the left action on the tangent bundle restricted to the subgroup H. Since $\lambda_h(o) = o$ we get that $\lambda_{h*}|_o : T_o(G/H) \to T_o(G/H)$ is an invertible operator. Differentiating (3.19) and (3.20) at e and h, respectively, yields

$$\pi_*|_h \circ L_{h*}|_e = \lambda_{h*}|_o \circ \pi_*|_e, \tag{3.22}$$
$$\pi_*|_e \circ R_{h^{-1}*}|_h = \pi_*|_h. \tag{3.23}$$

Combining these two equations together

$$\lambda_{h*}|_o \circ \pi_*|_e = \pi_*|_e \circ R_{h^{-1}*}|_h \circ L_{h*}|_e = \pi_*|_e \circ (R_{h^{-1}} \circ L_h)_*|_e = \pi_*|_e \circ \mathrm{Ad}_h, \tag{3.24}$$

we see that the action of H on $T_o(G/H)$ is realized via the adjoint representation on $\mathfrak{g}/\mathfrak{h}$. This observation together with the left action on $T(G/H)$ leads to the conclusion that one could hope to view the tangent bundle as an associated bundle

$$T(G/H) \simeq G \times_{\mathrm{Ad}_H} (\mathfrak{g}/\mathfrak{h}). \tag{3.25}$$

The isomorphism above is indeed true and it is a consequence of a more generic property that any vector bundle $p : E \to G/H$ equipped with a left action ℓ of G compatible with the natural left action λ on G/H in the sense that

$$p \circ \ell_g = \lambda_g \circ p, \quad g \in G, \tag{3.26}$$

corresponds to an associated bundle $G \times_\rho V \to G/H$ where V is the fiber over $o \equiv eH$ and ρ is the representation of H on V corresponding to the restriction of ℓ to H. This correspondence works in the following way [6, p.52]. Elements of $G \times_\rho V \equiv (G \times V)/H$ are orbits of pairs (g, v) under the right action $\mathfrak{R}$ of the group H

$$\mathfrak{R}_h(g, v) = (gh, \rho(h^{-1})v). \tag{3.27}$$

Therefore two pairs (g, v) and (g', v') belong to the same orbit if $g' = gh$ and $v' = \rho(h^{-1})v$ for some $h \in H$. We can map each orbit represented by (g, v) to an element of E using the left action ℓ

$$(g, v) \mapsto \ell_g(v). \tag{3.28}$$

This is independent of the choice of the pair. Choosing (g', v') we get

$$(g', v') \mapsto \ell_{g'}(v') = (\ell_g \circ \ell_h)(\rho(h^{-1})v) = (\ell_g \circ \ell_h \circ \ell_{h^{-1}})(v) = \ell_g(v). \tag{3.29}$$

For the other direction, consider $s \in E$ and choose an element $g \in G$ such that $\lambda_g(p(s)) = o$. The corresponding orbit in $(G \times V)/H$ is given by the pair $(g^{-1}, \ell_g(s))$. This is independent of the choice of g because for a different g' we get that $(g')^{-1}g \in H$ and so the pair $((g')^{-1}, \ell_{g'}(s))$ lies in the same orbit as $(g^{-1}, \ell_g(s))$.

Homogeneous space G/H can be viewed as a base manifold for principal H-bundle $\pi : G \to G/H$, where π is the quotient map. The principal action on the total space G is the right translation by the elements of the subgroup H. For such bundle some concepts are significantly simpler. For instance the fundamental vector field $\#X$, $X \in \mathfrak{h}$ at the point $g \in G$

$$\#X|_g = \left.\frac{d}{dt}\right|_{t=0} g \exp(tX) = L_{g*} \left.\frac{d}{dt}\right|_{t=0} \exp(tX) = L_{g*}(X) \tag{3.30}$$

is identical with the left-invariant vector field generated by X. On top of the principal action the bundle is naturally equipped with left action of G on itself via the left translation L_g, $g \in G$, that is compatible with the bundle projection and natural left action of G on G/H

$$\pi \circ L_g = \lambda_g \circ \pi. \tag{3.31}$$

The following theorem analyses G-invariant principal connections on this bundle [6, p.60]. These are invariant under the action of group G, which means that they are uniquely defined by their value at a single point

$$L_g^* A|_g = A|_e. \tag{3.32}$$

Theorem 3.1. *Let G be a Lie group and H its closed subgroup and $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras, respectively. There exists a G-invariant principal connection on $\pi : G \to G/H$, if and only if G/H is reductive, i.e. there exists an Ad_H-invariant decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}. \tag{3.33}$$

For a given decomposition the curvature F of the corresponding connection is also G-invariant and defined by its behaviour at the origin $\pi(e)$

$$\wedge^2 \mathfrak{m} \to \mathfrak{h} : \quad (X, Y) \mapsto -[X, Y]_{\mathfrak{h}}, \tag{3.34}$$

where the subscript denotes the $\mathfrak{h}$-part with respect to the chosen decomposition.

Proof. G-invariant $\mathfrak{h}$-valued one-form A satisfies $L_g^* A = A$ for any $g \in G$. This means that it is uniquely defined by behaviour at the unity $A|_e : \mathfrak{g} \to \mathfrak{h}$.

For A to be connection it must satisfy $A|_e(X) = X$ for any $X \in \mathfrak{h}$. Therefore one chooses $A|_e$ to be the projection of $\mathfrak{g}$ onto $\mathfrak{h}$ with respect to a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

Let A be a G-invariant principal connection on $\pi : G \to G/H$. We define the decomposition by setting $\mathfrak{m} = \ker A|_e$. This decomposition is Ad_H-invariant because $A|_e$ commutes with Ad_h for any $h \in H$

$$A|_e \circ \mathrm{Ad}_h = A|_e \circ R_{h^{-1}*} \circ L_{h*} = R_{h^{-1}}^* A|_h = \mathrm{Ad}_h \circ A|_e, \tag{3.35}$$

where in the second equality we used the defining relation for pullback of one-forms $f^*\omega = \omega \circ f_*$ and the fact that A is G-invariant, the third equality is the property of connections.

On the other hand let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a given Ad_H-invariant decomposition. We define a G-invariant one-form A by setting $A|_e$ to be the projection of $\mathfrak{g}$ onto $\mathfrak{h}$ and anywhere else to be

$$A|_g = L_{g^{-1}}^* A|_e. \tag{3.36}$$

The fundamental vector fields on $\pi : G \to G/H$ coincide with left-invariant vector fields on G so

$$A|_g(\#X) = A|_g(L_{g*}X) = A|_e \circ L_{g^{-1}*} \circ L_{g*}(X) = A|_e(X) = X, \quad X \in \mathfrak{h}. \tag{3.37}$$

The one-form A is right-equivariant because the decomposition is Ad_H-invariant

$$\begin{aligned} A|_{gh} &= A|_e \circ L_{(gh)^{-1}*} = A|_e \circ L_{h^{-1}*} \circ L_{g^{-1}*} = A|_e \circ L_{h^{-1}*} \circ R_{h*} \circ R_{h^{-1}*} \circ L_{g^{-1}*} \\ &= A|_e \circ \mathrm{Ad}_{h^{-1}} \circ R_{h^{-1}*} \circ L_{g^{-1}*} = \mathrm{Ad}_{h^{-1}} \circ A|_e \circ L_{g^{-1}*} \circ R_{h^{-1}*} \\ &= \mathrm{Ad}_{h^{-1}} \circ A|_g \circ R_{h^{-1}*}. \end{aligned}$$

If the connection A is G-invariant, then the curvature $F = dA + A \wedge A$ is also G-invariant because pullback commutes with the exterior derivative and product. Therefore F is defined by its behaviour at the unity $F|_e : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$

$$F|_e = dA|_e + A|_e \wedge A|_e. \tag{3.38}$$

We shall evaluate $dA|_e$ using the Cartan formula for exterior derivative

$$dA|_e(X,Y) = X^L(A(Y^L))|_e - Y^L(A(X^L))|_e - A|_e([X,Y]), \quad X, Y \in \mathfrak{g}, \tag{3.39}$$

where in the first two terms X^L and Y^L denote left-invariant vector fields corresponding to X and Y. The first two terms vanish because $A(Y^L) = Y$ is a constant function and therefore its derivative is zero everywhere. Finally we obtain

$$F|_e(X,Y) = -A|_e([X,Y]) + [A|_e(X), A|_e(Y)]. \tag{3.40}$$

The above expression is non-zero only if both entries belong to $\mathfrak{m}$, and in that case, the second commutator vanishes, and one obtains the desired map. □

This theorem is a corollary of a more generic theorem [6, p.57]

Theorem 3.2. *Let $i : H \to K$ be a homomorphism of Lie groups and consider principal K-bundle $P \to G/H$ equipped with left action of G compatible with the action on G/H. Then invariant principal connections on P are in one-to-one correspondence with linear maps $\alpha : \mathfrak{g} \to \mathfrak{k}$ such that*

1. $\alpha|_{\mathfrak{h}} = i_*|_e : \mathfrak{h} \to \mathfrak{k}$ *is the differential of i,*
2. $\alpha \circ \mathrm{Ad}_h = \mathrm{Ad}_{i(h)} \circ \alpha$ *for any $h \in H$.*

The idea of the proof is the same as above, however, there are some technical details.

From now on we restrict ourselves to the subclass of reductive homogeneous spaces which means that the Lie algebra $\mathfrak{g}$ of G admits Ad_H-invariant decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}. \tag{3.41}$$

That by the above theorem corresponds to fixing a connection on the principal bundle $\pi : G \to G/H$ and because of (3.25) the choice induces connection on the tangent bundle. However, it is instructive to examine this procedure in detail. For reductive homogeneous spaces, we can decompose the left-invariant Maurer-Cartan form on G as

$$\omega_{MC} = A + B, \tag{3.42}$$

where A is the connection form and B is a one-form with values in $\mathfrak{m}$. The behavior of B under the right action is inherited from the Maurer-Cartan form

$$R_h^*(\omega_{MC}) = \mathrm{Ad}_{h^{-1}}(\omega_{MC}) \quad \Rightarrow \quad R_h^*(B) = \mathrm{Ad}_{h^{-1}}(B),$$

for any $h \in H$. Furthermore, it is transverse to all vertical vectors because

$$B(\#X) = \omega_{MC}(\#X) - A(\#X) = X - X = 0, \quad X \in \mathfrak{h}. \tag{3.43}$$

These properties in the terminology of principal bundles make B into a horizontal one-form of type Ad. The form B in fact facilitates the isomorphism (3.25).

To see this let us fix a point $g \in G$. On the support of $B|_g$ the differential of π acts as an isomorphism therefore taking $V \in T_{\pi(g)}(G/H)$ and using the left-invariance inherited from the Maurer-Cartan form we get

$$\begin{aligned} B|_g \circ \pi_*|_g^{-1}(V) &= B|_e \circ L_{g^{-1}*}|_g \circ \pi_*|_g^{-1}(V) = B|_e \circ (\pi_*|_g \circ L_{g*}|_e)^{-1}(V) \\ &= B|_e \circ (\lambda_{g*}|_o \circ \pi_*|_e)^{-1}(V) = B|_e \circ \pi_*|_e^{-1} \circ \lambda_{g^{-1}*}|_{\pi(g)}(V), \end{aligned} \tag{3.44}$$

where we used the relation (3.22). The last expression sheds light onto the action of B. One first transports the tangent vector to the origin o via the left action of G on the tangent bundle and then assigns it the corresponding element of $\mathfrak{m}$. The corresponding orbit in $G \times_{\mathrm{Ad}_H} \mathfrak{m}$ is well defined thanks to the Ad-equivariance of B.

Lastly, it is interesting to examine the Maurer-Cartan equation with respect to the decomposition $\omega_{MC} = A + B$

$$0 = d\omega_{MC} + \frac{1}{2}[\omega_{MC} \wedge \omega_{MC}] = dA + \frac{1}{2}[A \wedge A] + \frac{1}{2}[B \wedge B] + dB + [A \wedge B]. \tag{3.45}$$

Taking the $\mathfrak{h}$ part and $\mathfrak{m}$ part we get

$$dA + \frac{1}{2}[A \wedge A] = -\frac{1}{2}[B \wedge B]_{\mathfrak{h}}, \tag{3.46}$$

$$dB + [A \wedge B] = -\frac{1}{2}[B \wedge B]_{\mathfrak{m}}, \tag{3.47}$$

where the term $[A \wedge B]$ belongs to $\mathfrak{m}$ due to the Ad_H invariance of decomposition (3.41). These equations look noticeably similar to equations (1.15) and (1.17) for the shape operator. The first says that the curvature is given by the commutator and the second says that the exterior covariant derivative of B amounts to the $\mathfrak{m}$-part of the commutator which is zero in the embedded case.

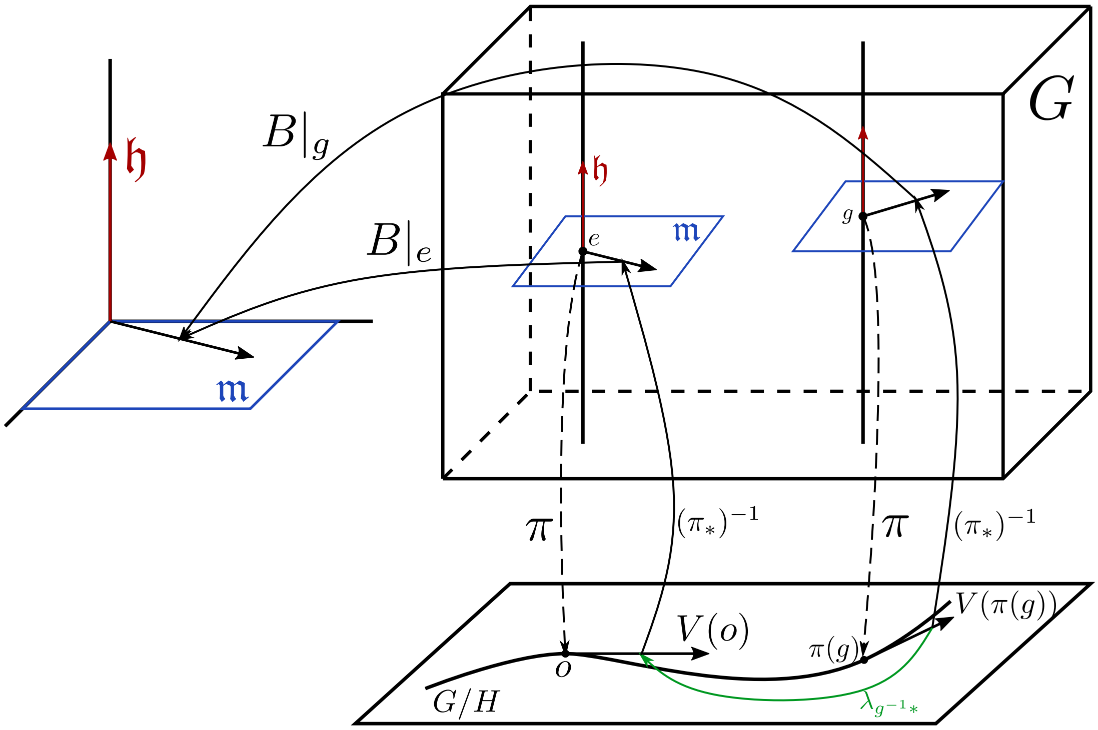

Figure 3.1: Visualization of the equation (3.44). Consequently, it shows the isomorphism of the tangent bundle and the associated bundle $G \times_{\mathrm{Ad}_H} \mathfrak{m}$ facilitated by the form B.

3.3 Universal connections

The idea behind universal connections is that one can find a special connection that gives rise to all other connections. This notion is encapsulated in the following theorem [18]

Theorem 3.3 (Narasimhan and Ramanan, 1962)**.** *Let G be a compact Lie group and d a positive integer. Then there exists a principal G-bundle B and a connection form A_0 on B such that any connection form on a principal G-bundle $p : P \to M$ with* $\dim M \leq d$ *is the pullback of A_0 by a principal bundle morphism of P to B.*

Since every compact group can be identified with a closed subgroup of some unitary group [18] the proof of this theorem boils down to proving this statement for G being a unitary group $U(n)$. In this case the bundle B is taken to be the complex Stiefel bundle $St_{\mathbb{C}}(n, N)$ over the complex Grassmannian $Gr_{\mathbb{C}}(n, N)$, where $N = (d+1)(2d+1)n^3$. The complex Stiefel manifold $St_{\mathbb{C}}(n, N)$ is the set of n-tuples of orthonormal vectors in $\mathbb{C}^N$ with respect to the standard scalar product. The complex Grassmannian $Gr_{\mathbb{C}}(n, N)$ is the set of n-dimensional subspaces of $\mathbb{C}^N$.

The elements of $St_{\mathbb{C}}(n, N)$ can be parametrised by matrices V with N rows and n orthonormal columns satisfying $\mathsf{V}^\dagger \mathsf{V} = \mathbb{1}$. On the Stiefel bundle there is a canonical connection A_0 which is a consequence of the generalized version of theorem 3.1 [6, p.57]. In coordinates the connection is given by

$$\mathsf{A}_0 = \mathsf{V}^\dagger d\mathsf{V}. \tag{3.48}$$

The main content of the proof is the search for the principal bundle morphism of a generic $U(n)$-bundle P to $St_{\mathbb{C}}(n, N)$. This has two parts. First, one solves the problem locally and then glues together a global map. The local problem can be summarized in the following way. First one finds $\mathbb{C}^{n,n}$-valued functions $\phi_1, \ldots \phi_{m'}$, $m' = (2d+1)n^2$, on the base manifold such that

$$\sum_{k=1}^{m'} \phi_k^\dagger \phi_k = \mathbb{1}_k, \quad \sum_{k=1}^{m'} \phi_k^\dagger d\phi_k = A, \tag{3.49}$$

where A is local connection form obtained from the bundle connection on P by a choice of gauge ξ_0. They present solution to this problem in an algebraic straightforward manner. However the choice of m' is not always optimal as we show later on in chapter 5. Next, one arranges these functions in a single matrix Φ with n columns and $m'' = nm'$ rows

$$\Phi(\xi) = \begin{pmatrix} \phi_1(p(\xi)) \\ \vdots \\ \phi_{m'}(p(\xi)) \end{pmatrix} \cdot \mathsf{R}, \quad \xi \in P \tag{3.50}$$

where the matrix R is given by $\xi = \xi_0(p(\xi)) \cdot \mathsf{R}$. This provides us with a local map from P to $St_{\mathbb{C}}(n, N)$. The global map is constructed using the resolution of unity.

Later they extended the theorem to connected Lie groups [19]. However, considerations employed there go beyond the scope of this thesis.

Let us compare this theorem with the Cartan-Janet theorem for isometric embedding [4, p.98]. In this setting, we have $n = d$ the dimension of the manifold. For m'' the Cartan-Janet theorem gives $m'' = d(d+1)/2$ whereas the universal connection theorem needs $m'' = (2d+1)d^3$. The Cartan-Janet truly achieves the same result. We can construct the desired map to the Stiefel manifold by constructing an orthonormal frame out of the derivatives $\partial_\mu f$ of the embedding f.

Chapter 4

Shape operator and rotating blade

We begin this chapter by recalling several basic concepts from the Yang-Mills theory [21, ch.6]. Assume we have a $U(n)$-theory, therefore a connection $i\mathsf{A}_\mu$ with values in algebra $\mathfrak{u}(n)$ of anti-hermitian matrices. The covariant derivative of a $\mathbb{C}^n$-valued field ψ is defined as

$$D_\mu \psi = \partial_\mu \psi + i\mathsf{A}_\mu \psi. \tag{4.1}$$

The gauge transformation given by $\mathsf{h} \in U(n)$ acts on the internal indices of the field $\psi \to \psi' = \mathsf{h}\psi$. In order to have $D'_\mu \psi' = \mathsf{h} D_\mu \psi$ the connection must transform as

$$\mathsf{A}'_\mu = \mathsf{h}\mathsf{A}_\mu \mathsf{h}^\dagger - i\mathsf{h}\partial_\mu \mathsf{h}^\dagger. \tag{4.2}$$

The measurable physical quantity is the field strength $\mathsf{F}_{\mu\nu}$ which is the curvature of connection $i\mathsf{A}_\mu$

$$\mathsf{F}_{\mu\nu} = -i[D_\mu, D_\nu] = \partial_\mu \mathsf{A}_\nu - \partial_\nu \mathsf{A}_\mu + i[\mathsf{A}_\mu, \mathsf{A}_\nu]. \tag{4.3}$$

The covariant derivative of $\mathsf{F}_{\mu\nu}$ (and any matrix for that matter) is defined as

$$D_\mu \mathsf{F}_{\nu\rho} = \partial_\mu \mathsf{F}_{\nu\rho} + i[\mathsf{A}_\mu, \mathsf{F}_{\nu\rho}]. \tag{4.4}$$

The field strength satisfies the following constraints called Bianchi identities

$$D_\mu \mathsf{F}_{\nu\rho} + D_\rho \mathsf{F}_{\mu\nu} + D_\nu \mathsf{F}_{\rho\mu} = 0, \tag{4.5}$$

which are automatically satisfied when $\mathsf{F}_{\mu\nu}$ is expressed using the gauge potential A_μ. The equations of motion for $\mathsf{F}_{\mu\nu}$

$$D_\mu \left(\sqrt{|g|} \mathsf{F}^{\mu\nu} \right) \tag{4.6}$$

can be obtained using the action principle for

$$\mathcal{S}_{YM}[\mathsf{A}_\mu] = -\frac{1}{4} \int_{M^d} d^d x \sqrt{|g|}\, \mathrm{Tr}(\mathsf{F}_{\mu\nu} \mathsf{F}^{\mu\nu}). \tag{4.7}$$

4.1 Introducing the rotating blade variable

In chapter 1 we discussed the shape operator in the context of submanifolds of Euclidean space. There the shape operator arises from the generalized Gauss map (*rotating blade*), which is essentially a map between the manifold and Grassmannian. The Grassmannian is a homogeneous space

$$Gr(n, N) \simeq O(N)/(O(n) \times O(N-n)) \tag{4.8}$$

and thus possesses the interesting properties laid out in the previous chapter. Our goal is to generalize the notion of a rotating blade and the shape operator to principal bundles. In particular, we are going to consider $U(n)$ bundles.

We are going to examine structures surrounding the following principal $U(n)$ bundle $p_1 : St_{\mathbb{C}}(n, N) \to Gr_{\mathbb{C}}(n, N)$, where $n < N$ and

$$St_{\mathbb{C}}(n, N) \simeq U(N)/U(N-n), \tag{4.9}$$

$$Gr_{\mathbb{C}}(n, N) \simeq U(N)/(U(n) \times U(N-n)) \tag{4.10}$$

are the complex equivalents of the Stiefel manifold and the Grassmannian, respectively. The complex Stiefel manifold is the set of n-tuples of vectors in $\mathbb{C}^N$ orthonormal in the sense of the standard scalar product. The complex Grassmannian is the set of n-dimensional subspaces of $\mathbb{C}^N$. The projection p_1 assigns to the n-tuple the subspace it spans. The elements of the Stiefel manifold can be understood as matrices with N rows and n orthonormal columns.

Consider two bundles over the Grassmannian. First is the aforementioned Stiefel bundle $p_1 : St_{\mathbb{C}}(n, N) \to Gr_{\mathbb{C}}(n, N)$, next is the "orthogonal complement" of the first $p_2 : St_{\mathbb{C}}(N-n, N) \to Gr_{\mathbb{C}}(n, N)$. The projection p_2 assigns to the $(N-n)$-tuple of vectors the orthogonal complement of their span.

As we discussed in chapter 2, the associated vector bundle to a principal bundle corresponding to a representation ρ can be modelled using functions of type ρ on the total space of the principal bundle. Now we assume ρ to be the defining representation of $U(n)$ and consider a function $\psi : St_{\mathbb{C}}(n, N) \to \mathbb{C}^n$ of type ρ. Using a local section of the Stiefel bundle $\mathsf{V} : Gr_{\mathbb{C}}(n, N) \to St_{\mathbb{C}}(n, N)$ we can lift ψ to $\Psi : Gr_{\mathbb{C}}(n, N) \to \mathbb{C}^N$ by setting

$$\Psi = \mathsf{V}\psi(\mathsf{V}). \tag{4.11}$$

If we chose a different section $\mathsf{V}' = \mathsf{V}\mathsf{h}$, where $\mathsf{h} \in U(n)$, then Ψ would not change

$$\Psi' = \mathsf{V}'\psi(\mathsf{V}') = \mathsf{V}\mathsf{h}\psi(\mathsf{V}\mathsf{h}) = \mathsf{V}\mathsf{h}\mathsf{h}^{-1}\psi(\mathsf{V}) = \Psi. \tag{4.12}$$

On the other hand, if we had a section Ψ of the trivial bundle such that at each point, it takes values in the subspace given by that point, then we could revert the above procedure to construct an equivariant function ψ with values in $\mathbb{C}^n$. Similarly, one could do this construction for $St_{\mathbb{C}}(N-n,N)$.

The Stiefel bundle possesses a canonical form of connection $i\mathsf{A}$ because the Killing form of $\mathfrak{u}(N)$ is negative-definite and thus gives a unique Ad-invariant complement of $\mathfrak{u}(n)$ in $\mathfrak{u}(N)$ (theorem 3.2). In coordinates given by matrix V, $\mathsf{V}^\dagger\mathsf{V}=\mathsf{1}$, the connection form reads

$$\mathsf{V}^\dagger d\mathsf{V} = i\mathsf{A}. \tag{4.13}$$

The covariant derivative of functions ψ transforming under the defining representation of $U(n)$ can then be written as

$$D\psi = d\psi + i\mathsf{A}\psi. \tag{4.14}$$

Interestingly, the same covariant derivative can be induced from a flat connection on the trivial bundle via

$$\mathsf{V}^\dagger d(\mathsf{V}\psi) = d\psi + \mathsf{V}^\dagger d\mathsf{V}\psi = d\psi + i\mathsf{A}\psi. \tag{4.15}$$

This goes similarly for $St_{\mathbb{C}}(N-n,N)$.

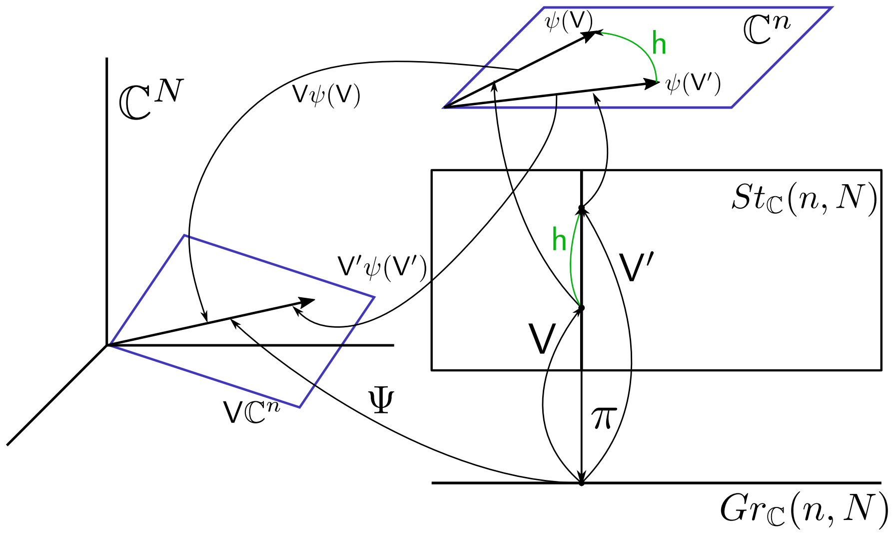

Figure 4.1: Cartoon illustrating the gauge-invariant lift described by equation (4.11). In green there is the gauge transformation $\mathsf{V}'=\mathsf{V}\mathsf{h}$ and the rotating blade is in blue.

This means that the direct sum of the two associated bundles can also be equipped with a connection that is the direct sum the two connections, however, thanks to the above discussion we can recast the result as a subbundle of the trivial $\mathbb{C}^N$ bundle over Grassmannian. The covariant derivative of a local $\mathbb{C}^N$-valued section Φ can be written in a compact form

$$\tilde{D}\Phi = \mathsf{P}_\parallel d(\mathsf{P}_\parallel \Phi) + \mathsf{P}_\perp d(\mathsf{P}_\perp \Phi) = d\Phi + i\mathsf{S}\Phi, \tag{4.16}$$

where $\mathsf{P}_\parallel$ is the orthogonal projector onto the subspace corresponding to the current point of $Gr_{\mathbb{C}}(n,N)$ and $\mathsf{P}_\perp$ projects onto the orthogonal complement. We call the operator $i\mathsf{S} = \mathsf{P}_\parallel \partial_\mu \mathsf{P}_\parallel + \mathsf{P}_\perp \partial_\mu \mathsf{P}_\perp$ the *shape operator*. One can alternatively write it as

$$i\mathsf{S} = \frac{1}{2}\mathsf{R}d\mathsf{R}, \tag{4.17}$$

where $\mathsf{R} = 2\mathsf{P}_\parallel - \mathsf{1}$ is the reflection with respect to the subspace $\mathsf{P}_\parallel$ projects onto. R satisfies $\mathsf{R}^2 = \mathsf{1}$ and we call it *rotating blade* in analogy with the geometry of embedded manifolds.

There are two ways of interpreting the shape operator. First we choose sections $\mathsf{V} : Gr_{\mathbb{C}}(n,N) \to St_{\mathbb{C}}(n,N)$ and $\mathsf{W} : Gr_{\mathbb{C}}(n,N) \to St_{\mathbb{C}}(N-n,N)$. This way we can write $\mathsf{P}_\parallel = \mathsf{V}\mathsf{V}^\dagger$ and $\mathsf{P}_\perp = \mathsf{W}\mathsf{W}^\dagger$ and thus

$$i\mathsf{S} = \mathsf{V}(\mathsf{V}^\dagger d\mathsf{V})\mathsf{V}^\dagger + \mathsf{W}(\mathsf{W}^\dagger d\mathsf{W})\mathsf{W}^\dagger + \mathsf{V}d\mathsf{V}^\dagger + \mathsf{W}d\mathsf{W}^\dagger. \tag{4.18}$$

The first two terms correspond to lifts of canonical connections, and the last two terms combine to give a lift of the left Maurer-Cartan form of $U(N)$ because the matrix (V,W) is unitary

$$(\mathsf{V},\mathsf{W})\left(\begin{pmatrix}\mathsf{V}^\dagger\\ \mathsf{W}^\dagger\end{pmatrix} d(\mathsf{V},\mathsf{W})\right)\begin{pmatrix}\mathsf{V}^\dagger\\ \mathsf{W}^\dagger\end{pmatrix} = -(\mathsf{V},\mathsf{W})d\begin{pmatrix}\mathsf{V}^\dagger\\ \mathsf{W}^\dagger\end{pmatrix} = -(\mathsf{V}d\mathsf{V}^\dagger + \mathsf{W}d\mathsf{W}^\dagger). \tag{4.19}$$

This way we can see S as a result of a $U(N)$ gauge transformation given by $\mathsf{U} = (\mathsf{V},\mathsf{W})$ which we call a *shape gauge*. Alternatively, using the decomposition

$$\mathfrak{u}(N) = \mathfrak{u}(n) \oplus \mathfrak{u}(N-n) \oplus \mathfrak{m} \tag{4.20}$$

the Maurer-Cartan form on $U(N)$ can be written as

$$\omega_{MC} = \begin{pmatrix} i\mathsf{A} & i\mathsf{B}^\dagger \\ i\mathsf{B} & i\mathsf{C}\end{pmatrix}, \tag{4.21}$$

where $\mathsf{A} \in \mathfrak{u}(n)$ and $\mathsf{C} \in \mathfrak{u}(N-n)$ are the canonical connections on $St_{\mathbb{C}}(n,N)$ and $St_{\mathbb{C}}(N-n,N)$, respectively and B is an arbitrary matrix with $N-n$ rows and n columns. Then one can rewrite

$$\mathsf{S} = -\mathsf{U}\begin{pmatrix} 0 & -\mathsf{B}^\dagger \\ \mathsf{B} & 0\end{pmatrix}\mathsf{U}^\dagger. \tag{4.22}$$

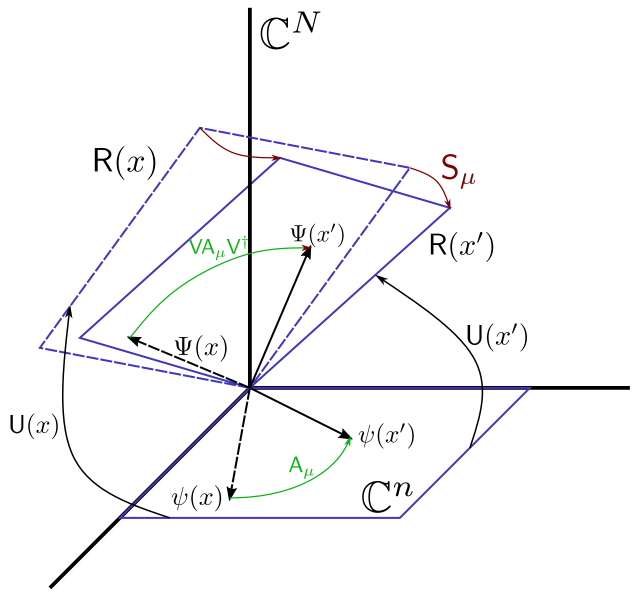

Figure 4.2: Illustration of the transformation to the shape gauge via the matrix $\mathsf{U} = (\mathsf{V}, \mathsf{W})$. It also attempts to showcase the effect of connection on the evolution of fields and how the rotating blade is evolving according to the shape operator.

In section 3.2 we showed that the form given by the off-diagonal matrix above plays a role of a solder form. It essentially represents the tangent vectors of the homogeneous space, in this case, the Grassmannian.

The combined curvature Ω of A and C

$$\Omega = \begin{pmatrix} \mathsf{F} & 0 \\ 0 & \mathsf{G} \end{pmatrix} \equiv \begin{pmatrix} d\mathsf{A} + \frac{i}{2}[\mathsf{A} \wedge \mathsf{A}] & 0 \\ 0 & d\mathsf{C} + \frac{i}{2}[\mathsf{C} \wedge \mathsf{C}] \end{pmatrix} \tag{4.23}$$

satisfies the equation (3.46) which in the shape gauge reads

$$\tilde{\Omega} = d\mathsf{S} + \frac{i}{2}[\mathsf{S} \wedge \mathsf{S}] = -\frac{i}{2}[\mathsf{S} \wedge \mathsf{S}]. \tag{4.24}$$

That is essentially the equation (1.17) we encountered in the geometry of embedded manifolds. The equation (1.15) can either be recovered via direct computation or from equation (3.47) by noticing that the off-diagonal part of the commutator of off-diagonal matrices always vanishes.

There are several advantages to this construction of the shape operator. First, it is canonical. Second, every step of the construction commutes with pullbacks. That means once there is a smooth map to the Grassmannian, one can pullback everything we constructed above, and the relations remain unchanged. That is why the theorem 3.3 of Narasimhan and Ramanan is so valuable.

Given a principal $U(n)$-bundle $p : P \to M$ with a connection $i\mathsf{A}_\mu$ we can for sufficiently high N find a map $\mathsf{V} : M \to St_{\mathbb{C}}(n, N)$ such that $i\mathsf{A}_\mu$ is the pullback of the canonical connection on Stiefel bundle

$$\mathsf{V}^\dagger \partial_\mu \mathsf{V} = i\mathsf{A}_\mu. \tag{4.25}$$

With V one can obtain the gauge invariant *rotating blade* $\mathsf{R} = 2\mathsf{V}\mathsf{V}^\dagger - \mathsf{1}$.

Let us illustrate this construction on a simple $U(1)$ gauge theory, the electromagnetism. Take for example the four-potential of an electromagnetic wave $A_\mu = n_\mu \sin(k_\nu x^\nu)$, where n_μ is the polarization vector and k_μ the four-momentum of the photon. Further we assume $N = 2$ so that

$$\mathsf{V} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cos\rho\, e^{i\alpha} \\ \sin\rho\, e^{i\beta} \end{pmatrix}, \quad |z_1|^2 + |z_2|^2 = 1. \tag{4.26}$$

Therefore we are supposed to find a solution to the equation

$$-i\mathsf{V}^\dagger \partial_\mu \mathsf{V} = \partial_\mu(\beta - \alpha)\sin^2\rho + \partial_\mu \alpha = n_\mu \sin(k_\nu x^\nu), \tag{4.27}$$

which is solved by $\alpha = -\beta = n_\mu x^\mu$ and $\rho = \frac{1}{2}(k_\nu x^\nu - \frac{\pi}{2})$. We obtain the following rotating blade

$$\mathsf{R} = \begin{pmatrix} \sin(k_\mu x^\mu) & -\cos(k_\mu x^\mu) e^{2i(n_\nu x^\nu)} \\ -\cos(k_\mu x^\mu) e^{-2i(n_\nu x^\nu)} & -\sin(k_\mu x^\mu) \end{pmatrix}. \tag{4.28}$$

This simple example also sheds light on possible inconveniences of this approach. The solution V is not unique. We can generate other solutions via multiplication from the left by a unitary matrix U, which must satisfy

$$\mathsf{V}^\dagger \mathsf{U}^\dagger (\partial_\mu \mathsf{U}) \mathsf{V} = 0. \tag{4.29}$$

For instance constant unitary matrix yields another solution. However, even in the case of equation (4.27) there is a whole one-parameter family of non-constant matrices

$$\mathsf{V}_\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ -b_\gamma & a^*_\gamma \end{pmatrix} \begin{pmatrix} \sin(k_\mu x^\mu) \\ -\cos(k_\mu x^\mu) \end{pmatrix}, \tag{4.30}$$

where

$$a_\gamma = \cos(\gamma n_\mu x^\mu) + \frac{i}{\gamma}\sin(\gamma n_\mu x^\mu), \quad b_\gamma = \frac{\sqrt{\gamma^2 - 1}}{\gamma}\sin(\gamma n_\mu x^\mu). \tag{4.31}$$

The presented solution is recovered by setting $\gamma = 1$. It is apparent that if we chose $N > 2$ we could get even more solutions to the equation (4.27) by simply doing

$$\mathsf{V} = \frac{\cos(f(x))}{\sqrt{2}} \begin{pmatrix} \sin(k_\mu x^\mu)\, e^{in_\nu x^\nu} \\ -\cos(k_\mu x^\mu)\, e^{-in_\nu x^\nu} \\ 0 \\ 0 \end{pmatrix} + \frac{\sin(f(x))}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ \sin(k_\mu x^\mu)\, e^{in_\nu x^\nu} \\ -\cos(k_\mu x^\mu)\, e^{-in_\nu x^\nu} \end{pmatrix}, \tag{4.32}$$

where $f(x)$ is arbitrary function.

Nevertheless, it is still interesting to encode the connection using the matrix V. The gauge transformation of the connection translates to right multiplication by the corresponding group element

$$(\mathsf{V}\mathsf{h})^\dagger \partial_\mu (\mathsf{V}\mathsf{h}) = \mathsf{h}^\dagger (\mathsf{V}^\dagger \partial_\mu \mathsf{V}) \mathsf{h} + \mathsf{h}^\dagger \partial_\mu \mathsf{h} = i(\mathsf{h}^\dagger \mathsf{A}_\mu \mathsf{h} - i\mathsf{h}^\dagger \partial_\mu \mathsf{h}), \tag{4.33}$$

Furthermore, the covariant derivative of a $\mathbb{C}^n$-valued fields

$$D_\mu \psi = \partial_\mu \psi + i\mathsf{A}_\mu \psi \tag{4.34}$$

can be lifted via V to act on $\mathbb{C}^N$ in a gauge-invariant manner

$$\mathsf{V}\mathsf{V}^\dagger \partial_\mu (\mathsf{V}\psi) = \mathsf{V}\partial_\mu \psi + \mathsf{V}(\mathsf{V}^\dagger \partial_\mu \mathsf{V})\psi = \mathsf{V}(\partial_\mu \psi + i\mathsf{A}_\mu \psi) = \mathsf{V} D_\mu \psi. \tag{4.35}$$

We know that this constitutes a part of larger connection on $\mathbb{C}^N$-valued fields Φ defined as

$$\tilde{D}_\mu \Phi = \mathsf{P}_\parallel \partial_\mu (\mathsf{P}_\parallel \Phi) + \mathsf{P}_\perp \partial_\mu (\mathsf{P}_\perp \Phi) = \partial_\mu \Phi + i\mathsf{S}_\mu \Phi, \tag{4.36}$$

where S_μ is the shape operator

$$\mathsf{P}_\parallel = \mathsf{V}\mathsf{V}^\dagger, \quad \mathsf{P}_\perp = 1 - \mathsf{V}\mathsf{V}^\dagger, \tag{4.37}$$

$$i\mathsf{S}_\mu = \mathsf{P}_\parallel \partial_\mu \mathsf{P}_\parallel + \mathsf{P}_\perp \partial_\mu \mathsf{P}_\perp = \frac{1}{2} \mathsf{R} \partial_\mu \mathsf{R}. \tag{4.38}$$

The field strength associated with connection A_μ

$$\mathsf{F}_{\mu\nu} = -i[D_\mu, D_\nu] = \partial_\mu \mathsf{A}_\nu - \partial_\nu \mathsf{A}_\mu + i[\mathsf{A}_\mu, \mathsf{A}_\nu] \tag{4.39}$$

is recovered from the curvature of shape operator

$$\tilde{\Omega}_{\mu\nu} = \partial_\mu \mathsf{S}_\nu - \partial_\nu \mathsf{S}_\mu + i[\mathsf{S}_\mu, \mathsf{S}_\nu] = -i[\mathsf{S}_\mu, \mathsf{S}_\nu] = -\frac{i}{4}[\partial_\mu \mathsf{R}, \partial_\nu \mathsf{R}] \tag{4.40}$$

via the following formula

$$\mathsf{F}_{\mu\nu} = \mathsf{V}^\dagger \tilde{\Omega}_{\mu\nu} \mathsf{V}. \tag{4.41}$$

Interestingly, the Bianchi identities

$$D_\mu \mathsf{F}_{\nu\rho} + D_\rho \mathsf{F}_{\mu\nu} + D_\nu \mathsf{F}_{\rho\mu} = 0 \tag{4.42}$$

for the shape gauge curvature feature only partial derivative due to the Jacobi identity for the commutator

$$\partial_\mu \tilde{\Omega}_{\nu\rho} + \partial_\rho \tilde{\Omega}_{\mu\nu} + \partial_\nu \tilde{\Omega}_{\rho\mu} = 0. \tag{4.43}$$

4.2 Dynamics of the rotating blade

In the previous section we have introduced the variable R, the rotating blade. It would be interesting to formulate dynamical equations for this variable. We start with the usual vacuum Yang-Mills action. It only includes the curvature squared term

$$\mathcal{S}_{YM}[\mathsf{A}_\mu] = -\frac{1}{4}\int_{M^d} d^dx\sqrt{|g|}\,\mathrm{Tr}(\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}), \tag{4.44}$$

where g is the determinant of the metric on spacetime M^d. We can rewrite this in terms of the shape gauge curvature $\tilde{\Omega}_{\mu\nu}$ using the relation

$$\mathsf{V}^\dagger\tilde{\Omega}_{\mu\nu}\mathsf{V} = \mathsf{F}_{\mu\nu}, \tag{4.45}$$

where V is some choice of gauge giving $\mathsf{R} = 2\mathsf{V}\mathsf{V}^\dagger - 1$. Utilising the cyclic property of the trace we finally arrive at an action formulated in terms of the rotating blade R

$$\mathcal{S}_{YM}[\mathsf{R}] = -\frac{1}{4}\int_{M^d} d^dx\sqrt{|g|}\,\mathrm{Tr}(\tilde{\Omega}_{\mu\nu}\tilde{\Omega}^{\mu\nu}\mathsf{P}), \tag{4.46}$$

where P is the projector on eigenspace of R associated with $+1$. Interestingly, this action only includes the first derivatives of R because

$$\tilde{\Omega}_{\mu\nu} = -\frac{i}{4}[\partial_\mu\mathsf{R}, \partial_\nu\mathsf{R}]. \tag{4.47}$$

The variations in R are done slightly differently than usual. We consider the following variations

$$\mathsf{R} \to e^{i\delta\mathsf{X}}\mathsf{R}e^{-i\delta\mathsf{X}} \approx \mathsf{R} + \underbrace{i[\delta\mathsf{X}, \mathsf{R}]}_{\delta\mathsf{R}}, \tag{4.48}$$

where $\delta\mathsf{X}$ has values in hermitian matrices. We consider these variations as they do not change the relation $\mathsf{R}^2 = 1$ and R remains hermitian. Before calculating the variation of action, we calculate few intermediate results

$$\delta\mathsf{P} = \frac{1}{2}\delta(2\mathsf{P}-1) = \frac{1}{2}\delta\mathsf{R} = \frac{i}{2}[\delta\mathsf{X}, \mathsf{R}] = i[\delta\mathsf{X}, \mathsf{P}], \tag{4.49}$$

$$\delta\tilde{\Omega}_{\mu\nu} = -\frac{i}{4}\delta\left([\partial_\mu\mathsf{R}, \partial_\nu\mathsf{R}]\right) = \left[\delta\mathsf{X}, \tilde{\Omega}_{\mu\nu}\right] + \frac{1}{4}\left[[\partial_\mu\delta\mathsf{X}, \mathsf{R}], \partial_\nu\mathsf{R}\right] - \frac{1}{4}\left[[\partial_\nu\delta\mathsf{X}, \mathsf{R}], \partial_\mu\mathsf{R}\right]. \tag{4.50}$$

We also note the following relations

$$\mathrm{Tr}([A,B]C) = \mathrm{Tr}(A[B,C]) \tag{4.51}$$

$$[\mathsf{P}, \tilde{\Omega}_{\mu\nu}] = -\frac{i}{8}[\mathsf{R}, [\partial_\mu\mathsf{R}, \partial_\nu\mathsf{R}]] = 0. \tag{4.52}$$

The variation of action has three terms

$$\delta\mathcal{S}_{YM} = -\frac{1}{4}\int_{M^d} d^d x\sqrt{|g|}\,\mathrm{Tr}(\delta\tilde{\Omega}_{\mu\nu}\tilde{\Omega}^{\mu\nu}\mathsf{P} + \tilde{\Omega}_{\mu\nu}\delta\tilde{\Omega}^{\mu\nu}\mathsf{P} + \tilde{\Omega}_{\mu\nu}\tilde{\Omega}^{\mu\nu}\delta\mathsf{P}). \tag{4.53}$$

The last term vanishes

$$\mathrm{Tr}(\tilde{\Omega}_{\mu\nu}\tilde{\Omega}^{\mu\nu}\delta\mathsf{P}) = \mathrm{Tr}(\tilde{\Omega}_{\mu\nu}\tilde{\Omega}^{\mu\nu}i[\delta\mathsf{X},\mathsf{P}]) = \mathrm{Tr}(i\delta\mathsf{X}\,[\mathsf{P},\tilde{\Omega}_{\mu\nu}\tilde{\Omega}^{\mu\nu}]) = 0. \tag{4.54}$$

Using the cyclic property of the trace and the fact that the projector commutes with curvature the remaining part can be rearranged to give

$$\delta\mathcal{S}_{YM} = -\frac{1}{2}\int_{M^d} d^d x\sqrt{|g|}\,\mathrm{Tr}\left(\delta\tilde{\Omega}_{\mu\nu}\tilde{\Omega}^{\mu\nu}\mathsf{P}\right) \tag{4.55}$$

$$= -\frac{1}{4}\int_{M^d} d^d x\sqrt{|g|}\,\mathrm{Tr}\left([[\partial_\mu\delta\mathsf{X},\mathsf{R}]\,,\partial_\nu\mathsf{R}]\,\tilde{\Omega}^{\mu\nu}\mathsf{P}\right) \tag{4.56}$$

$$= -\frac{1}{4}\int_{M^d} d^d x\sqrt{|g|}\,\mathrm{Tr}\left(\partial_\mu\delta\mathsf{X}\left[\mathsf{R},\left[\partial_\nu\mathsf{R},\tilde{\Omega}^{\mu\nu}\mathsf{P}\right]\right]\right), \tag{4.57}$$

because

$$\mathrm{Tr}\left(\left[\delta\mathsf{X},\tilde{\Omega}_{\mu\nu}\right]\tilde{\Omega}^{\mu\nu}\mathsf{P}\right) = \mathrm{Tr}\left(\delta\mathsf{X}\left[\tilde{\Omega}_{\mu\nu},\tilde{\Omega}^{\mu\nu}\mathsf{P}\right]\right) = 0. \tag{4.58}$$

After performing integration by parts and demanding the stationarity condition we arrive at the expression

$$0 = \delta\mathcal{S}_{YM} = \frac{1}{4}\int_{M^d} d^d x\,\mathrm{Tr}\left(\delta\mathsf{X}\;\partial_\mu\left[\mathsf{R},\left[\partial_\nu\mathsf{R},\tilde{\Omega}^{\mu\nu}\mathsf{P}\right]\right]\right), \tag{4.59}$$

which is satisfied when

$$\partial_\mu\left[\mathsf{R},\left[\partial_\nu\mathsf{R},\tilde{\Omega}^{\mu\nu}\mathsf{P}\right]\right] = 0. \tag{4.60}$$

It is possible to simplify this even more by using the Jacobi identity

$$\left[\mathsf{R},\left[\partial_\nu\mathsf{R},\tilde{\Omega}^{\mu\nu}\mathsf{P}\right]\right] = \left[[\mathsf{R},\partial_\nu\mathsf{R}]\,,\tilde{\Omega}^{\mu\nu}\mathsf{P}\right] + \left[\partial_\nu\mathsf{R},\left[\mathsf{R},\tilde{\Omega}^{\mu\nu}\mathsf{P}\right]\right] = 4\left[\mathsf{S}_\nu,\sqrt{|g|}\tilde{\Omega}^{\mu\nu}\mathsf{P}\right].$$

Using the properties of the shape operator one finally recovers

$$\begin{aligned}\partial_\mu\left[\mathsf{S}_\nu,\sqrt{|g|}\tilde{\Omega}^{\mu\nu}\mathsf{P}\right] &= \frac{1}{2}\left[\partial_\mu\mathsf{S}_\nu - \partial_\nu\mathsf{S}_\mu,\sqrt{|g|}\tilde{\Omega}^{\mu\nu}\mathsf{P}\right] + \left[\mathsf{S}_\nu,\partial_\mu\left(\sqrt{|g|}\tilde{\Omega}^{\mu\nu}\mathsf{P}\right)\right] \\ &= -i\left[[\mathsf{S}_\mu,\mathsf{S}_\nu]\,,\sqrt{|g|}\tilde{\Omega}^{\mu\nu}\mathsf{P}\right] + \left[\mathsf{S}_\nu,\partial_\mu\left(\sqrt{|g|}\tilde{\Omega}^{\mu\nu}\mathsf{P}\right)\right] \\ &= \left[\mathsf{S}_\nu,\partial_\mu\left(\sqrt{|g|}\tilde{\Omega}^{\mu\nu}\mathsf{P}\right)\right].\end{aligned} \tag{4.61}$$

The partial divergence can actually be replaced by the covariant one to yield the following equation of motion

$$\left[\mathsf{S}_\nu, \tilde{D}_\mu\left(\sqrt{|g|}\tilde{\Omega}^{\mu\nu}\mathsf{P}\right)\right] = 0. \tag{4.62}$$

To make contact with the original theory, one can perform a gauge transformation given by $\mathsf{U}^\dagger = (\mathsf{V},\mathsf{W})^\dagger$ to obtain

$$\left[\begin{pmatrix} 0 & -B_\nu^\dagger \\ B_\nu & 0 \end{pmatrix}, \begin{pmatrix} D_\mu\left(\sqrt{|g|}\mathsf{F}^{\mu\nu}\right) & 0 \\ 0 & 0 \end{pmatrix}\right] = 0. \tag{4.63}$$

The conclusion is that for any R such that corresponding $\mathsf{F}_{\mu\nu}$ is a vacuum solution, the equation above is also satisfied. However, the equation above admits non-vacuum solutions where the sources J^ν must satisfy

$$\mathsf{B}_\nu\mathsf{J}^\nu + \mathsf{J}^\nu\mathsf{B}_\nu^\dagger = 0, \tag{4.64}$$

which features the matrix-valued form B which decomposes (3.46) both the original and emergent curvature and thus couples them implicitly together.

Apart from the Yang-Mills action one could consider a simpler action, where instead of contracting curvatures, one contracts shape operators. This turns out to be the action of non-linear sigma model

$$\mathcal{S}_\sigma[\mathsf{R}] = \int_{M^d} d^dx\sqrt{|g|}g^{\mu\nu}\,\mathrm{Tr}(\mathsf{S}_\mu\mathsf{S}_\nu) = -\frac{1}{4}\int_{M^d} d^dx\sqrt{|g|}g^{\mu\nu}\,\mathrm{Tr}(\partial_\mu\mathsf{R}\,\partial_\nu\mathsf{R}), \tag{4.65}$$

where the target space is the complex Grassmannian [14, 20, 3]. We perform the variations in similar manner as above

$$\mathsf{R} \to e^{i\delta\mathsf{X}}\,\mathsf{R}\,e^{-i\delta\mathsf{X}} \approx \mathsf{R} + i[\delta\mathsf{X},\mathsf{R}]. \tag{4.66}$$

The variation of this action is much simpler than the Yang-Mills action. The resulting equations of motion read

$$\partial_\mu\left(\sqrt{|g|}\mathsf{S}^\mu\right) = 0. \tag{4.67}$$

We compare these equations of motion with Maxwell equations in section 5.1.

Chapter 5

Examples

5.1 Electromagnetism

The electromagnetic theory can be formulated in terms of $U(1)$ connection iA_μ on a four-dimensional manifold [2, pt.I][9, ch.16]. The group $U(1)$ is abelian and consists of elements of the form $e^{i\theta}$. The gauge transformation of the four-potential A_μ then reads

$$A'_\mu = e^{-i\theta} A_\mu e^{i\theta} - ie^{-i\theta}\partial_\mu e^{i\theta} = A_\mu + \partial_\mu \theta. \tag{5.1}$$

To obtain the universal connection one should find $\mathsf{V} \in U(N)/U(N-1)$, where N is sufficiently large. We begin with observations regarding $N = 2$. Elements of $U(2)/U(1)$ are two-component unit vectors which can be parametrised by three real parameters α, β and ρ as

$$\mathsf{V} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cos\rho\; e^{i\alpha} \\ \sin\rho\; e^{i\beta} \end{pmatrix}, \quad |z_1|^2 + |z_2|^2 = 1. \tag{5.2}$$

In this case, the equation for the pullback of the canonical connection reads

$$-i\mathsf{V}^\dagger \partial_\mu \mathsf{V} = \partial_\mu \alpha\; \cos^2\rho + \partial_\mu\beta\; \sin^2\rho = A_\mu. \tag{5.3}$$

The corresponding electromagnetic field reads

$$F = d(\alpha - \beta) \wedge d(\sin^2\rho), \tag{5.4}$$

which satisfies the following relation $F \wedge F = \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = 0$. This is equivalent with saying that the electric and magnetic field are orthogonal to each other [17, sec.25]. This is not the most generic electromagnetic field, however following observations will later be used to improve the algorithm for finding matrix V for a generic electromagnetic field.

The complementary matrices W, i.e. sections of the "orthogonal" bundle, are parametrised by a real parameter θ

$$\mathsf{W} = \begin{pmatrix} -z_2^* \\ z_1^* \end{pmatrix} e^{i\theta}. \tag{5.5}$$

The complementary emergent connection C_μ evaluates to

$$C_\mu = \mathsf{W}^\dagger \partial_\mu \mathsf{W} = -(A_\mu - \partial_\mu \theta), \tag{5.6}$$

which corresponds to the same electromagnetic field coupling to an opposite charge

$$G_{\mu\nu} \equiv \partial_\mu C_\nu - \partial_\mu C_\nu + i[C_\mu, C_\nu] = -F_{\mu\nu}. \tag{5.7}$$

The rotating blade $\mathsf{R} = 2\mathsf{V}\mathsf{V}^\dagger - 1$ in this parametrization yields

$$\mathsf{R} = \begin{pmatrix} \cos 2\rho & e^{i(\alpha-\beta)} \sin 2\rho \\ e^{-i(\alpha-\beta)} \sin 2\rho & -\cos 2\rho \end{pmatrix}. \tag{5.8}$$

The corresponding shape operator is easily calculated from $\mathsf{S}_\mu = \frac{1}{2}\mathsf{R}\partial_\mu \mathsf{R}$

$$\mathsf{S}_\mu = \begin{pmatrix} -\sin 2\rho & e^{i(\alpha-\beta)} \cos 2\rho \\ e^{-i(\alpha-\beta)} \cos 2\rho & \sin 2\rho \end{pmatrix} \frac{\sin 2\rho}{2} \partial_\mu(\alpha-\beta) + \begin{pmatrix} 0 & -ie^{i(\alpha-\beta)} \\ ie^{-i(\alpha-\beta)} & 0 \end{pmatrix} \partial_\mu \rho. \tag{5.9}$$

The shape gauge curvature $\tilde{\Omega}_{\mu\nu} = -i[\mathsf{S}_\mu, \mathsf{S}_\nu]$ coincidentally takes the form

$$\tilde{\Omega}_{\mu\nu} = \begin{pmatrix} \cos 2\rho & e^{i(\alpha-\beta)} \sin 2\rho \\ e^{-i(\alpha-\beta)} \sin 2\rho & -\cos 2\rho \end{pmatrix} (\partial_\mu(\alpha-\beta)\partial_\nu\rho - \partial_\nu(\alpha-\beta)\partial_\mu\rho) \sin 2\rho. \tag{5.10}$$

In this setup the equations of motion for the rotating blade (4.62) allow for non-vacuum solutions satisfying

$$\partial_\mu F^{\mu\nu}(B_\nu + B_\nu^*) = 0, \tag{5.11}$$

where $B_\nu = -i\mathsf{W}^\dagger \partial_\nu \mathsf{V}$. This equation admits linear solutions $\alpha - \beta = 2n_\mu x^\mu$ and $\rho = \frac{1}{2}(k_\mu x^\mu - \frac{\pi}{2})$ which correspond to a wave-like four-potential (in $\alpha + \beta = 0$ gauge)

$$A_\mu = n_\mu \sin(k_\nu x^\nu), \tag{5.12}$$

where k_μ and n_μ satisfy $(n_\mu k_\nu - n_\nu k_\mu)(n^\mu k^\nu - n^\nu k^\mu) = 0$. This condition includes the usual electromagnetic wave with polarization n and four-momentum k, however one could consider a situation where the character of n and k is swapped and the equations would still hold. The resulting sources

$$J^\nu = \partial_\mu F^{\mu\nu} = 2\cos(2\rho)\, k_\mu k^\mu n^\nu \tag{5.13}$$

would have light-like character.

The relationship between the equations of motion (4.67) given by the non-linear sigma model and Maxwell equations is unclear. We get two equations for the two gauge invariant parameters ρ and $\alpha - \beta$

$$\partial_\mu \partial^\mu (4\rho) = \sin(4\rho)\, \partial_\mu (\alpha - \beta) \partial^\mu (\alpha - \beta), \tag{5.14}$$

$$\sin(4\rho)\, \partial_\mu \partial^\mu (\alpha - \beta) = -(1 + \cos(4\rho)) \partial_\mu (4\rho) \partial^\mu (\alpha - \beta). \tag{5.15}$$

This system admits linear solutions $\rho = k_\nu x^\nu$, $\alpha - \beta = n_\nu x^\nu$ if the four-vector n is light-like and orthogonal to k. However, this choice leads straightforwardly to the four-potential of an electromagnetic planar wave (4.27)

$$-i\mathsf{V}^\dagger \partial_\mu \mathsf{V} = 2\partial_\mu(\alpha - \beta)\, \cos(2\rho) + 2\partial_\mu(\alpha + \beta) = 2n_\mu \cos(2k_\nu x^\nu) + 2\partial_\mu(\alpha + \beta), \tag{5.16}$$

where this is a solution of the vacuum Maxwell equations if k is light-like and orthogonal to n, however, this is satisfied only when k is collinear with n, which amounts to a pure gauge potential and therefore no field.

5.1.1 Illustrative example: Magnetic monopole

Let us illustrate these tools on a notoriously famous system with interesting geometry, the Dirac monopole. The setup is the following. Assume there is a radial magnetic field $\vec{B} = \frac{g}{r^3}\vec{r}$ given by a vector potential $\vec{A}$. The flux of such field through a sphere of radius R is $4\pi g$. If there was a globally defined vector potential, one could use Stoke's theorem to calculate the flux as the integral over the sphere's boundary. The sphere's boundary is an empty set, and thus the flux would be zero. That leads to a conclusion that the vector potential must be singular somewhere [22].

To describe this situation one is forced to the use bundles [24] and local potentials $\vec{A}^+$ and $\vec{A}^-$ whose corresponding one-forms have the following components in the usual spherical coordinates (r, θ, φ) [10, p.444]

$$A^+_\varphi = g(1 - \cos\theta), \quad A^+_r = A^+_\theta = 0, \tag{5.17}$$

$$A^-_\varphi = -g(1 + \cos\theta), \quad A^-_r = A^-_\theta = 0, \tag{5.18}$$

where A^+ is defined on the region $\theta \neq \pi$ and A^- on the region $\theta \neq 0$. On the overlap they are related via a gauge transformation

$$h = \exp(2ig\varphi)\,. \tag{5.19}$$

It is illustrative to find matrix V for both potentials. Therefore, we have two equations

$$-i\mathsf{V}_+^\dagger \partial_\varphi \mathsf{V}_+ = \cos^2\rho_+\, \partial_\varphi \alpha_+ + \sin^2\rho_+\, \partial_\varphi \beta_+ = 2g\sin^2(\theta/2) \tag{5.20}$$

$$-i\mathsf{V}_-^\dagger \partial_\varphi \mathsf{V}_- = \cos^2\rho_-\, \partial_\varphi \alpha_- + \sin^2\rho_-\, \partial_\varphi \beta_- = -2g\cos^2(\theta/2). \tag{5.21}$$

The solutions read

$$\mathsf{V}_+ = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\, e^{2ig\varphi} \end{pmatrix} = \begin{pmatrix} \cos(\theta/2)\, e^{-2ig\varphi} \\ \sin(\theta/2) \end{pmatrix} e^{2ig\varphi} = \mathsf{V}_- h. \tag{5.22}$$

The matrices V_+ and V_- carry the same singularities as the original potentials, however, the rotating blade is consistently defined everywhere including the problematic z-axis ($\{\theta = 0\} \cup \{\theta = \pi\}$) without the origin ($\{r = 0\}$)

$$\mathsf{R} = \begin{pmatrix} \cos\theta & \sin\theta\; e^{-2ig\varphi} \\ \sin\theta\; e^{2ig\varphi} & -\cos\theta \end{pmatrix}. \tag{5.23}$$

5.1.2 Generic electromagnetic field

For a generic electromagnetic field on a d-dimensional manifold it is sufficient to consider locally $\mathsf{V} \in U(N)/U(N-1)$ with at most $N = 2\lfloor \frac{d}{2} \rfloor$. The A_μ can be cast into one of the following normal forms [4, p.40]

$$A = \phi_0 d\phi_1 + \phi_2 d\phi_3 + \ldots + \phi_{2r} d\phi_{2r+1}, \tag{5.24}$$

$$A = d\phi_1 + \phi_2 d\phi_3 + \ldots + \phi_{2r} d\phi_{2r+1}, \tag{5.25}$$

where r is defined as being the least number such that

$$A \wedge (dA)^r \equiv A \wedge \underbrace{dA \wedge \ldots \wedge dA}_{r \text{ times}} \neq 0, \quad A \wedge (dA)^{r+1} = 0. \tag{5.26}$$

The construction of V is straightforward. Similarly, as for the electromagnetic wave or the Dirac monopole, each term in the sum above will be given by a two-component complex vector V_k of length $\frac{1}{r}$. The resulting V is obtained by stacking these into a single column vector.

The two-component vector V_k can be parametrised by three real parameters

$$\mathsf{V}_k = \frac{1}{\sqrt{r}} \begin{pmatrix} \cos\rho_k\; e^{i\alpha_k} \\ \sin\rho_k\; e^{i\beta_k} \end{pmatrix}. \tag{5.27}$$

Let us denote $\lambda_k = \sup|\phi_k|$ then we solve

$$\mathsf{V}_k^\dagger d\mathsf{V}_k = \frac{2}{r}\left(d(\alpha_k + \beta_k) + \cos(2\rho_k) d(\alpha_k - \beta_k)\right) = \frac{\phi_k}{\lambda_k} \lambda_k d\phi_{k+1} \tag{5.28}$$

by setting $\alpha_k + \beta_k = \text{const.}$, $\alpha_k - \beta_k = \frac{r\lambda_k}{2}\phi_{k+1}$ and $\cos(2\rho_k) = \frac{\phi_k}{\lambda_k}$. The maximal possible r is $\lfloor \frac{d}{2} \rfloor$ because for any higher r the corresponding exterior power is automatically zero. This approach gives a better lower bound on N than in [18], where Narasimhan and Ramanan give the lower bound $N \geq 2d + 1$ in the case of electromagnetism.

5.2 Non-Abelian Yang-Mills theories

There is still a lot of work remaining regarding the generic Yang-Mills theories. At this time, the only known method of finding the matrix V was outlined by Narasimhan and Ramanan in [18]. In this section we offer an improved version of the first step in their construction. First, one writes out the gauge potential A in components

$$\mathsf{A} = \sum_{a=1}^{n^2} \sum_{\mu=1}^{d} A_\mu^a \mathsf{E}_a dx^\mu, \tag{5.29}$$

where E_a is the basis of $n \times n$ hermitian matrices made of elements of this form

$$\begin{aligned}
(\mathsf{E}_a)_{kl} &= \delta_{ka}\delta_{al},\ a = 1, \dots, n \\
(\mathsf{E}_a)_{kl} &= (\mathsf{K}_{ij})_{kl} = \frac{1}{2}\left(\delta_{ki}\delta_{il} + \delta_{kj}\delta_{jl} + \delta_{ki}\delta_{jl} + \delta_{kj}\delta_{il}\right),\ a = n+1, \dots, n + \binom{n}{2} \\
(\mathsf{E}_a)_{kl} &= (\mathsf{L}_{ij})_{kl} = \frac{1}{2}\left(\delta_{ki}\delta_{il} + \delta_{kj}\delta_{jl} - i\delta_{ki}\delta_{jl} + i\delta_{kj}\delta_{il}\right),\ a = n + \binom{n}{2} + 1, \dots, n^2.
\end{aligned} \tag{5.30}$$

All of these matrices are positive semidefinite and satisfy $\mathsf{E}_a^2 = \mathsf{E}_a$. Then we can solve n^2 equations

$$\mathsf{V}_a^\dagger \partial_\mu \mathsf{V}_a = i A_\mu^a, \tag{5.31}$$

which are of the same form as for a generic electromagnetic field. This means that it is sufficient for every column vector V_a to have $m' = 2\lfloor \frac{d}{2} \rfloor$ rows. Now one can form the following (Kronecker) tensor product

$$\mathsf{V}_a \otimes \mathsf{E}_a, \tag{5.32}$$

which produces a matrix with $m'n = 2\lfloor \frac{d}{2} \rfloor n$ rows and n columns. The matrix is obtained essentially by plugging the matrix E_a into the slots of the column vector V_a. The desired matrix V is obtained by stacking matrices $\mathsf{V}_a \otimes \mathsf{E}_a$ on top of each other to finally obtain a matrix with $m'' = 2\lfloor \frac{d}{2} \rfloor n^3$ rows and n columns. This is an improvement over the original $m'' = (2d+1)n^3$ [18]. To verify

$$\mathsf{V}^\dagger \partial_\mu \mathsf{V} = \sum_{a=1}^{n^2} (\mathsf{V}_a^\dagger \partial_\mu \mathsf{V}_a) \otimes \left(\mathsf{E}_a^\dagger \mathsf{E}_a\right) = \sum_{a=1}^{n^2} i A_\mu^a \otimes \mathsf{E}_a = i\mathsf{A}_\mu. \tag{5.33}$$

Let us illustrate this approach for group $SU(2)$. We can view it as a subgroup of $U(2)$ and therefore its algebra is a subalgebra of $\mathfrak{u}(2)$. An $SU(2)$ gauge potential A_μ can be decomposed as

$$\mathsf{A}_\mu = A_\mu^1 \sigma_1 + A_\mu^2 \sigma_2 + A_\mu^3 \sigma_3, \tag{5.34}$$

where σ_a are the usual Pauli matrices. To use the outlined procedure we need to rewrite A_μ in the basis (5.30) which in this case reads

$$\mathsf{E}_1 = \frac{1+\sigma_3}{2},\ \mathsf{E}_2 = \frac{1-\sigma_3}{2},\ \mathsf{E}_3 = \frac{1+\sigma_1}{2},\ \mathsf{E}_4 = \frac{1+\sigma_2}{2}. \tag{5.35}$$

In this basis the gauge potential reads

$$\mathsf{A}_\mu = (A^3_\mu - A^1_\mu - A^2_\mu)\mathsf{E}_1 - (A^1_\mu + A^2_\mu + A^3_\mu)\mathsf{E}_2 + 2A^1_\mu\mathsf{E}_3 + 2A^2_\mu\mathsf{E}_4. \tag{5.36}$$

Now we need to solve four equations

$$-i\mathsf{V}_1^\dagger\partial_\mu\mathsf{V}_1 = A^3_\mu - A^1_\mu - A^2_\mu, \tag{5.37}$$

$$-i\mathsf{V}_2^\dagger\partial_\mu\mathsf{V}_2 = -(A^1_\mu + A^2_\mu + A^3_\mu), \tag{5.38}$$

$$-i\mathsf{V}_3^\dagger\partial_\mu\mathsf{V}_3 = 2A^1_\mu, \tag{5.39}$$

$$-i\mathsf{V}_4^\dagger\partial_\mu\mathsf{V}_4 = 2A^2_\mu, \tag{5.40}$$

so that the we can form the desired matrix V

$$\mathsf{V} = \begin{pmatrix} \mathsf{V}_1 \otimes \mathsf{E}_1 \\ \mathsf{V}_2 \otimes \mathsf{E}_2 \\ \mathsf{V}_3 \otimes \mathsf{E}_3 \\ \mathsf{V}_4 \otimes \mathsf{E}_4 \end{pmatrix}. \tag{5.41}$$

5.3 Gravity

Einstein's theory of gravity assumes that spacetime is not a static object, and its evolution is influenced by its content. Gravity is then an apparent force that results from the non-trivial curvature of spacetime. More technically, spacetime is a Lorentzian manifold, i.e., manifold equipped with a metric tensor $g_{\mu\nu}$ of signature $(d,1)$, where usually $d=3$. Alternatively, we can say that it locally resembles the Minkowski spacetime. The geometry then evolves (in geometrized units $G=c=1$) according to equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi\mathcal{T}_{\mu\nu}, \tag{5.42}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor, R the Ricci scalar and $\mathcal{T}_{\mu\nu}$ the energy-momentum tensor. In the following discussion we are going to assume that our universe is empty, which makes the right-hand side of the equations zero

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \tag{5.43}$$

These equations are supposed to be understood as equations for the metric tensor $g_{\mu\nu}$, and when written out explicitly, they are second-order non-linear partial differential equations. They can also be obtained from the action principle for the Einstein-Hilbert action

$$\mathcal{S}[g] = \int_M R\sqrt{|g|}dx, \tag{5.44}$$

where g is the determinant of the metric tensor.

There is an alternative variational approach called Palatini formalism. We follow the approach in [2, ch.3]. In this one assumes that the metric is not the fundamental field and instead one takes a *frame* field e^a_μ which diagonalizes the metric

$$g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu, \tag{5.45}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ is the usual Minkowski metric. One can also regard e as a 1-form with values in the Minkowski space. This form is sometimes called a *solder* form.

The second ingredient of Palatini formalism is the connection. However, it is not the usual affine connection on the tangent bundle. The point of Palatini formalism is to leave the tangent space via the frame field and do most of the work on the Minkowski space. Therefore we define connection A by prescribing covariant derivative of sections of the Minkowski bundle over our manifold

$$(D_\mu v)^a = \partial_\mu v^a + A_\mu{}^a{}_b v^b. \tag{5.46}$$

We further demand the connection to be metric, which means

$$\partial_\mu(\eta_{ab}v^a w^b) = \eta_{ab}(D_\mu v)^a w^b + \eta_{ab}v^a(D_\mu w)^b. \tag{5.47}$$

As a result the matrices $\mathsf{A}_\mu = (A_\mu{}^a{}_b)$ must satisfy

$$\mathsf{A}_\mu^T\eta + \eta\mathsf{A}_\mu = 0. \tag{5.48}$$

These are the defining relations for the Lie algebra of the Lorentz group. Then using the frame and the connection one defines as in [2, p.408] the *imitation Levi-Civita connection* on the manifold

$$\hat{\Gamma}^\rho{}_{\mu\nu} = g^{\rho\sigma}\eta_{ab}e^a_\sigma A_\mu{}^b{}_c e^c_\nu = g^{\rho\sigma}\eta_{ab}e^a_\sigma(D_\mu e_\nu)^b = g^{\rho\sigma}e^T_\sigma\,\eta\,D_\mu e_\nu. \tag{5.49}$$

The bundle curvature $\mathsf{F} = d\mathsf{A} + \frac{1}{2}[\mathsf{A}\wedge\mathsf{A}]$ is used to define the associated *imitation Riemann tensor*

$$\hat{R}^\rho{}_{\sigma\mu\nu} = g^{\rho\kappa}\,\eta_{ab}\,e^a_\kappa\,F_{\mu\nu}{}^b{}_c e^c_\sigma, \tag{5.50}$$

which is used to define the *imitation Ricci tensor* $\hat{R}_\mu\nu = \hat{R}^\rho{}_{\mu\rho\nu}$ and the *imitation Ricci scalar* $\hat{R} = \hat{R}^\mu{}_\mu$. The Palatini action is then formulated in terms of the frame e and connection A

$$\mathcal{S}[e, A] = \int_M \hat{R}\sqrt{|g|}dx = \int_M \sqrt{|g|}e^\mu_a e^\nu_b F_{\mu\nu}{}^{ab}. \tag{5.51}$$

The variations of the frame yield the Einstein equations for the imitation variables

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}g_{\mu\nu} = 0, \tag{5.52}$$

whereas variations of the connection give the following condition [21, p.216]

$$D_\mu \left(\sqrt{|g|}(e^\nu (e^\mu)^T - e^\mu (e^\nu)^T)\right) = 0, \tag{5.53}$$

which can be equivalently formulated as saying that the imitation Levi-Civita connection coincides with the Levi-Civita connection derived from the metric $g_{\mu\nu}$.

Our approach to Yang-Mills's theory can also be applied here. We can encode the bundle connection A_μ into a matrix $\mathsf{V} = (V_a^I)$ via

$$\mathsf{V}^T \eta \partial_\mu \mathsf{V} = \mathsf{A}_\mu. \tag{5.54}$$

Then we can lift the frame e_μ^a to a gauge-invariant frame

$$f_\mu^I = (\mathsf{V}e_\mu)^I = V_a^I\, e_\mu^a. \tag{5.55}$$

This is similar to the approach in [8]. This way we can express the bundle curvature $\mathsf{F}_{\mu\nu}$ using the shape gauge curvature $\tilde{\Omega}_{\mu\nu}$

$$\eta \mathsf{F}_{\mu\nu} = \mathsf{V}^T \tilde{\eta} \tilde{\Omega}_{\mu\nu} \mathsf{V}, \tag{5.56}$$

where $\tilde{\eta}$ is the N-dimensional analogue of η. This way the imitation Ricci scalar reads

$$\hat{R} = g^{\mu\nu} g^{\rho\sigma} e_\mu^T \mathsf{V}^T \tilde{\eta} \tilde{\Omega}_{\nu\rho} \mathsf{V} e_\sigma = g^{\mu\nu} g^{\rho\sigma} f_\mu^T \tilde{\eta} \tilde{\Omega}_{\nu\rho} f_\sigma. \tag{5.57}$$

Now we replace the connection in the Palatini action with the matrix V

$$\mathcal{S}[e, \mathsf{V}] = \int_M \hat{R}\sqrt{|g|}dx = \int_M \sqrt{|g|} e_a^\mu e_b^\nu (\mathsf{V}^T \tilde{\eta} \tilde{\Omega}_{\mu\nu} \mathsf{V})^{ab}. \tag{5.58}$$

Since the dependence on the frame remains unchanged the variations of the frame yield again the Einstein equations.

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}g_{\mu\nu} = f_\rho^T \eta [\partial^\rho \mathsf{R}, \partial_\nu \mathsf{R}] f_\mu - \frac{1}{2} f_\mu^T \eta f_\nu \; f_\rho^T \eta [\partial^\rho \mathsf{R}, \partial^\sigma \mathsf{R}] f_\sigma = 0, \tag{5.59}$$

where $\mathsf{R} = 2\mathsf{V}\mathsf{V}^\dagger - \mathsf{1}$ is the rotating blade. Note that the equations contains only first derivative of R. For the variations of V it is advantageous to rewrite the action using matrix trace

$$\mathcal{S}[e, \mathsf{V}] = \int_M \sqrt{|g|}\, \mathrm{Tr}\left((e^\mu)^T \mathsf{V}^T \tilde{\eta} \tilde{\Omega}_{\mu\nu} \mathsf{V} e^\nu\right). \tag{5.60}$$

We are going to use essentially the same variations we used in Yang-Mills theory

$$\mathsf{V} \to e^{\delta \mathsf{X}}\mathsf{V} \approx \mathsf{V} + \underbrace{\delta \mathsf{X}\,\mathsf{V}}_{\delta \mathsf{V}}, \tag{5.61}$$

where $\delta\mathsf{X}$ is matrix from the Lie algebra of Lorentz group. The variation of action has three terms which can be under the trace rearranged into

$$\delta\mathcal{S} = \int_M \sqrt{|g|}\,\mathrm{Tr}\left(\mathsf{V}e^\nu(\mathsf{V}e^\mu)^T\tilde{\eta}\left(\delta\tilde{\Omega}_{\mu\nu} - [\delta\mathsf{X},\tilde{\Omega}_{\mu\nu}]\right)\right). \tag{5.62}$$

These variations in V give the same variations of the rotating blade $\mathsf{R} = 2\mathsf{V}\mathsf{V}^T\tilde{\eta} - 1$ we were considering in the Yang-Mills case. Thus we can use the result (4.50) for the variation of curvature

$$\delta\mathcal{S} = \frac{1}{4}\int_M \sqrt{|g|}\,\mathrm{Tr}\left(\left(f^\nu(f^\mu)^T\tilde{\eta} - f^\mu(f^\nu)^T\tilde{\eta}\right)[[\partial_\mu\delta\mathsf{X},\mathsf{R}],\partial_\nu\mathsf{R}]\right). \tag{5.63}$$

Using similar manipulations as in the Yang-Mills case and realizing that

$$[\mathsf{R}, f^\nu(f^\mu)^T\tilde{\eta} - f^\mu(f^\nu)^T\tilde{\eta}] = 0$$

one arrives at the following equation

$$0 = \partial_\mu\left[\mathsf{S}_\nu, \sqrt{|g|}(f^\nu(f^\mu)^T\eta - f^\mu(f^\nu)^T\eta)\right]. \tag{5.64}$$

Using the property of the shape operator

$$\partial_\mu\mathsf{S}_\nu - \partial_\nu\mathsf{S}_\mu = -2[\mathsf{S}_\mu,\mathsf{S}_\nu], \tag{5.65}$$

and the Jacobi identity, the equation (5.64) can be rewritten as

$$\left[\mathsf{S}_\nu, \tilde{D}_\mu\left(\sqrt{|g|}(f^\nu(f^\mu)^T\eta - f^\mu(f^\nu)^T\eta)\right) + \left[\mathsf{S}_\mu, \sqrt{|g|}(f^\nu(f^\mu)^T\eta - f^\mu(f^\nu)^T\eta)\right]\right] = 0. \tag{5.66}$$

This equation is noticeably different from (5.53), however, Faddeev shows [8] that connection with vanishing torsion, i.e.,

$$\hat{\Gamma}^\rho{}_{\mu\nu} = \hat{\Gamma}^\rho{}_{\nu\mu}, \tag{5.67}$$

should satisfy the equation (5.66). However, he admits, that it is not the only possibility and that the equation could be satisfied by connections and frames with non-trivial torsion.

Conclusion

In this thesis, we set out to find a place for the shape operator in the context of gauge theories. In the first chapter, we noticed that we could use the shape operator to describe the Levi-Civita connection once the manifold was embedded. In the third chapter, we learned about homogeneous spaces and the vital theorem 3.3 of Narasimhan and Ramanan about universal connections. The theorem essentially says that any connection can be obtained from the canonical connection on the Stiefel bundle.

That led us in the fourth chapter to investigate the geometry surrounding the Stiefel bundle over Grassmannian. We found there exists a canonical object with the same properties as the shape operator encountered in the first chapter. Moreover, this canonical shape operator swallows any apriori present gauge freedom. It is gauge-independent and allows gauge-independent treatment of both gauge fields and matter fields (4.11). The relation to the shape operator of embedded geometry also became apparent. The shape operator of an embedded manifold is just the pullback of the canonical one under the Gauss map. In the fifth chapter, we attempted to make contact with traditional theory. Additionally, we improved the dimensional requirements for constructing the universal connection.

This thesis is, however, just a beginning. Many aspects of this approach are still unknown. There are technical questions such as choosing optimal dimensions for the universal connection or the ambiguity of the solution. In the case of Levi-Civita connections, these questions are answered by the embedding theorems. However, the answer eludes us in the case of generic principal bundles.

Then there are the conceptual questions. In the fourth chapter, we introduced the rotating blade, a generalization of the Gauss map from embedded geometry. Could we promote the rotating blade to be a fundamental variable? We showed that, yes (4.46), one could formulate the action in terms of the rotating blade. However, the resulting equations of motion (4.62) were richer and contained the original field equations only as a particular case. Nevertheless, the added benefit of the rotating blade is that it is gauge-invariant and may be more suitable for quantization, where gauge freedom causes inconvenience in both path integral [21, ch.7] and canonical [11, sec.3-2] approach.

The rotating blade also appears to have better behavior than the gauge potentials. When we visited the example of the Dirac monopole, we had to use only locally defined gauge potentials. However, we found that the rotating blade was defined everywhere and had no singularities. It would be interesting to investigate field configurations of non-abelian gauge theories to examine the behavior of the rotating blade in those situations.

Another interesting application could be in the study of coupled systems. The rotating blade defines the shape operator, which is a direct sum of two connections (4.16). These are coupled via the rotating blade. We have seen this in the case of the electromagnetic field (5.7), where the rotating blade provides the connection for both positively and negatively charged particles.

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