# Jordan-Schwinger map in the theory of angular momentum

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The Jordan-Schwinger representation of the su(2) algebra utilizes ladder operators to efficiently handle su(2) representations with arbitrary spin. In these notes we point out the usefulness of this technique for calculating the Clebsch-Gordan coefficients when two angular momenta are being composed.

## I. JORDAN-SCHWINGER REPRESENTATION: GENERIC CASE

Let the $n \times n$ matrices $\mathbb{A}_1, \ldots, \mathbb{A}_N$ form a representation (typically fundamental) of a Lie algebra $\mathfrak{g}$:

$$[\mathbb{A}_i, \mathbb{A}_j] = c_{ij}^k \mathbb{A}_k, \tag{1}$$

where $c_{ij}^k$ are the structure constants of $\mathfrak{g}$, and summation over $k = 1, \ldots, N$ is implied.

Consider $n$ pairs of creation and annihilation operators $\hat{a}_1^\dagger, \ldots, \hat{a}_n^\dagger$ and $\hat{a}_1, \ldots, \hat{a}_n$ with usual bosonic commutation relations

$$[\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta} \quad , \quad [\hat{a}_\alpha, \hat{a}_\beta] = 0 \quad , \quad [\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger] = 0. \tag{2}$$

Then, the operators

$$\hat{A}_i = \hat{a}_\alpha^\dagger (\mathbb{A}_i)_{\alpha\beta} \hat{a}_\beta, \tag{3}$$

where $(\mathbb{A}_i)_{\alpha\beta}$ is the $(\alpha, \beta)$-th entry of the matrix $\mathbb{A}_i$, and summation over $\alpha, \beta = 1, \ldots, n$ is implied, form again a representation of the Lie algebra $\mathfrak{g}$. Indeed,

$$\begin{aligned}
[\hat{A}_i, \hat{A}_j] &= (\mathbb{A}_i)_{\alpha\beta} (\mathbb{A}_j)_{\gamma\delta}\, [\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger \hat{a}_\delta] \\
&= (\mathbb{A}_i)_{\alpha\beta} (\mathbb{A}_j)_{\gamma\delta}\, (\hat{a}_\alpha^\dagger [\hat{a}_\beta, \hat{a}_\gamma^\dagger] \hat{a}_\delta + \hat{a}_\gamma^\dagger [\hat{a}_\alpha^\dagger, \hat{a}_\delta] \hat{a}_\beta) \\
&= (\mathbb{A}_i)_{\alpha\beta} (\mathbb{A}_j)_{\beta\delta}\, \hat{a}_\alpha^\dagger \hat{a}_\delta - (\mathbb{A}_i)_{\alpha\beta} (\mathbb{A}_j)_{\gamma\alpha}\, \hat{a}_\gamma^\dagger \hat{a}_\beta \\
&= (\mathbb{A}_i \mathbb{A}_j - \mathbb{A}_j \mathbb{A}_i)_{\alpha\delta}\, \hat{a}_\alpha^\dagger \hat{a}_\delta \\
&= c_{ij}^k (\mathbb{A}_k)_{\alpha\beta} \hat{a}_\alpha^\dagger \hat{a}_\beta \\
&= c_{ij}^k \hat{A}_k.
\end{aligned} \tag{4}$$

The map $\mathbb{A}_i \mapsto \hat{A}_i$ from matrices to operators (on an abstract Hilbert space) is referred to as the Jordan-Schwinger map [1].

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## II. JORDAN-SCHWINGER REPRESENTATION: $\mathfrak{su}(2)$

In particular, we will be concerned with the angular momentum Lie algebra $\mathfrak{su}(2)$, whose fundamental representation is spanned by the $2 \times 2$ matrices $\mathbb{J}_i = \frac{\sigma_i}{2}$, $i = 1, 2, 3$, which fulfil the commutation relations

$$[\mathbb{J}_i, \mathbb{J}_j] = i\, \varepsilon_{ijk}\, \mathbb{J}_k. \tag{5}$$

Here $\sigma_i$ denote the standard Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

For the case of $\mathfrak{su}(2)$, the Jordan-Schwinger map yields the following operators:

$$\begin{aligned} \hat{J}_1 &= \frac{1}{2}(\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2), \\ \hat{J}_2 &= \frac{i}{2}(\hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2), \\ \hat{J}_3 &= \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2). \end{aligned} \tag{7}$$

From now on we shall omit the 'hat', writing simply $J_i$ instead of $\hat{J}_i$, and $a_\alpha$ instead of $\hat{a}_\alpha$.

The angular momentum ladder operators $J_\pm = J_1 \pm iJ_2$ assume a particularly simple form

$$J_+ = a_1^\dagger a_2 \quad , \quad J_- = a_1 a_2^\dagger. \tag{8}$$

The angular momentum squared, $\vec{J}^2 = J_1^2 + J_2^2 + J_3^2$, reads

$$\begin{aligned} \vec{J}^2 &= J_3^2 + \frac{1}{2}(J_+J_- + J_-J_+) \\ &= \frac{1}{4}(a_1^\dagger a_1 - a_2^\dagger a_2)^2 + \frac{1}{2}(a_1^\dagger a_1\, a_2 a_2^\dagger + a_2^\dagger a_2\, a_1 a_1^\dagger) \\ &= \frac{1}{4}(N_1^2 - 2N_1N_2 + N_2^2) + \frac{1}{2}(N_1N_2 - N_1 + N_1N_2 - N_2) \\ &= \frac{N}{2}\left(\frac{N}{2} + 1\right), \end{aligned} \tag{9}$$

where, in passing, we have denoted by $N_1$, $N_2$, and $N$ the number operators

$$N_1 = a_1^\dagger a_1 \quad , \quad N_2 = a_2^\dagger a_2. \quad , \quad N = N_1 + N_2. \tag{10}$$

(Note that now $J_3 = \frac{1}{2}(N_1 - N_2)$.)

Normalized states with occupation numbers $n_1$, $n_2$ (i.e., the simultaneous eigenstates of operators $N_1$, $N_2$) read

$$|n_1, n_2\rangle = \frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}} |0\rangle\,, \tag{11}$$

where $|0\rangle$ is the abstract vacuum state, and $n_1, n_2 = 0, 1, 2, \ldots$. These are also eigenstates $|j, m\rangle$ of $J_3$ and $\vec{J}^2$:

$$J_3\, |j,m\rangle = m\, |j,m\rangle \quad , \quad \vec{J}^2\, |j,m\rangle = j(j+1) \quad , \quad |j,m\rangle = |n_1 = j+m, n_2 = j-m\rangle\,. \tag{12}$$

Therefore, adding a ‘quantum’ with $a_1^\dagger$ increases the spin $j$ and the spin projection $m$ by $\frac{1}{2}$, whereas adding a ‘quantum’ with $a_2^\dagger$ increases $j$ by $\frac{1}{2}$, but decreases $m$ by $\frac{1}{2}$ (see Fig.).

Note that the operators $J_i$ of Eq. (7) preserve the total occupation number $n_1+n_2$, and hence also the value of spin $j$. The Fock space generated by $a_1^\dagger$, $a_2^\dagger$ then decomposes into subspaces, labelled by $j=\frac{1}{2}(n_1+n_2)=0,\frac{1}{2},1,\ldots$, which are invariant under the $J_i$ (and, of course, under the derived operators $J_\pm$ and $\vec{J}^2$). Moreover, within the spin-$j$ subspace, $m=-j,-j-1,\ldots,j$, as follows from the inequalities $j+m=n_1\geq 0$ and $j-m=n_2\geq 0$.

Let us remark that one can realize the abstract Fock space as a space of functions in two complex variables $f(z_1,z_2)$, and the abstract creation and annihilation operators as multiplicative and differential operators

$$a_1^\dagger \simeq z_1 \quad , \quad a_2^\dagger \simeq z_2 \quad , \quad a_1 \simeq \frac{\partial}{\partial z_1} \quad , \quad a_2 \simeq \frac{\partial}{\partial z_2}. \tag{13}$$

The vacuum state $|0\rangle$ is identified with 1, and the scalar product can be defined via a two-fold integral over the complex plane

$$\langle f|g\rangle = \frac{1}{\pi^2}\int_{\mathbb{C}^2} f^*(z_1,z_2)\, g(z_1,z_2)\, e^{-|z_1|^2-|z_2|^2} dz_1 dz_1^* dz_2 dz_2^*. \tag{14}$$

The states $|j,m\rangle$ are then realized by the polynomials

$$|j,m\rangle \simeq \frac{z_1^{j+m}}{\sqrt{(j+m)!}}\frac{z_2^{j-m}}{\sqrt{(j-m)!}}. \tag{15}$$

### A. Spin coherent states

Let us define a spin coherent state by the formula

$$|j,\mu\rangle = e^{\mu J_-}\,|j,m=j\rangle \quad , \quad \mu\in\mathbb{C}. \tag{16}$$

$$\begin{aligned}
|j,\mu\rangle &= e^{\mu a_2^\dagger a_1}\frac{(a_1^\dagger)^{2j}}{\sqrt{(2j)!}}\,|0\rangle \\
&= e^{\mu a_2^\dagger a_1}\frac{(a_1^\dagger)^{2j}}{\sqrt{(2j)!}}e^{-\mu a_2^\dagger a_1}\,|0\rangle \\
&= \frac{1}{\sqrt{(2j)!}}(e^{\mu a_2^\dagger a_1}a_1^\dagger e^{-\mu a_2^\dagger a_1})^{2j}\,|0\rangle \\
&= \frac{(a_1^\dagger+\mu a_2^\dagger)^{2j}}{\sqrt{(2j)!}}\,|0\rangle\,.
\end{aligned} \tag{17}$$

To prove the last equality, observe that

$$\frac{d}{d\mu}(e^{\mu a_2^\dagger a_1}a_1^\dagger e^{-\mu a_2^\dagger a_1}) = a_2^\dagger\, e^{\mu a_2^\dagger a_1}[a_1,a_1^\dagger]e^{-\mu a_2^\dagger a_1} = a_2^\dagger. \tag{18}$$

In passing we note that Eq. (17) implies

$$\frac{(J_-)^\ell}{\ell!}\,|j,j\rangle = \frac{1}{\sqrt{(2j)!}}\binom{2j}{\ell}(a_1^\dagger)^{2j-\ell}(a_2^\dagger)^\ell\,|0\rangle = \binom{2j}{\ell}^{1/2}|j,j-\ell\rangle\,. \tag{19}$$

## III. COMPOSITION OF TWO ANGULAR MOMENTA

The Jordan-Schwinger representation for a system of two independent angular momenta (labelled $a$ and $b$) utilizes four pairs of creation and annihilation operators, and identifies

$$J_i^a = \frac{1}{2}(\sigma_i)_{\alpha\beta} a_\alpha^\dagger a_\beta \quad , \quad J_i^b = \frac{1}{2}(\sigma_i)_{\alpha\beta} b_\alpha^\dagger b_\beta \quad , \quad J_i^{tot} = J_i^a + J_i^b. \tag{20}$$

(Explicit expressions are analogous to those of Eq. (7).) Since $[J_i^a, J_j^b] = 0$ for all $i, j = 1, 2, 3$, the composed angular momentum operators $J_i^{tot}$ satisfy the $\mathfrak{su}(2)$ commutation relations, Eq. (5).

Our task is now to build out of the tensor product states $|j_a, m_a\rangle\, |j_b, m_b\rangle = |j_a, m_a\rangle \otimes |j_b, m_b\rangle$ (i.e., eigenstates of the operators $(\vec{J}^a)^2, J_3^a, (\vec{J}^b)^2, J_3^b$) linear combinations that are eigenstates of operators $(\vec{J}^a)^2, (\vec{J}^b)^2, (\vec{J}^{tot})^2, J_3^{tot}$. We shall denote the latter states by $|j_a, j_b, j_{tot}, m_{tot}\rangle$, and look for their expansion in terms of $|j_a, m_a\rangle\, |j_b, m_b\rangle$. The coefficients in this expansion are the Clebsch-Gordan coefficients.

First, we realize that

$$|j_a, j_b, j_a + j_b, j_a + j_b\rangle = |j_a, j_a\rangle\, |j_b, j_b\rangle = \frac{(a_1^\dagger)^{2j_a}}{\sqrt{(2j_a)!}} \frac{(b_1^\dagger)^{2j_b}}{\sqrt{(2j_b)!}}\, |0\rangle \tag{21}$$

are common eigenstates for both sets of operators. From these we will generate all the other eigenstates $|j_a, j_b, j_{tot}, m_{tot}\rangle$ using the ladder operator $J_-^{tot}$, which lowers the eigenvalue $m_{tot}$, and the operator [2]

$$S^\dagger = a_2^\dagger b_1^\dagger - a_1^\dagger b_2^\dagger, \tag{22}$$

which fulfils the following commutation relations:

$$[N^{a,b}, S^\dagger] = N^{a,b} \quad , \quad [J_3^{tot}, S^\dagger] = 0 \quad , \quad [J_\pm^{tot}, S^\dagger] = 0. \tag{23}$$

Moreover, by the last two relations, and the first line in Eq. (9),

$$[(\vec{J}^{tot})^2, S^\dagger] = 0. \tag{24}$$

That is, the operator $S^\dagger$ raises (simultaneously) the eigenvalues $j_a$ and $j_b$ by $\frac{1}{2}$, while preserving $j_{tot}$ and $m_{tot}$:

$$S^\dagger\, |j_a, j_b, j_{tot}, m_{tot}\rangle = \alpha\, \left|j_a + \tfrac{1}{2}, j_b + \tfrac{1}{2}, j_{tot}, m_{tot}\right\rangle. \tag{25}$$

To determine the factors $\alpha$, we realize that the operator $SS^\dagger$ can be cast as

$$SS^\dagger = \left(\frac{N^a + N^b}{2} + 1\right)\left(\frac{N^a + N^b}{2} + 2\right) - (\vec{J}^{tot})^2. \tag{26}$$

Hence, choosing $\alpha$ real and positive, we find

$$\begin{aligned} \alpha(j_a, j_b, j_{tot}) &= \sqrt{(j_a + j_b + 1)(j_a + j_b + 2) - j_{tot}(j_{tot} + 1)} \\ &= \sqrt{(j_a + j_b + 1 - j_{tot})(j_a + j_b + 2 + j_{tot})}, \end{aligned} \tag{27}$$

and after repeated application of $S^\dagger$ we obtain

$$\frac{(S^\dagger)^k}{k!}\, |j_a, j_b, j_a + j_b, m_{tot}\rangle = \binom{2(j_a + j_b) + k + 1}{k}^{1/2} \left|j_a + \tfrac{k}{2}, j_b + \tfrac{k}{2}, j_a + j_b, m_{tot}\right\rangle. \tag{28}$$

Now, Eqs. (28) and (19) give (writing for the moment $j'_a, j'_b$ instead of $j_a, j_b$)

$$\frac{(J_-^{tot})^\ell}{\ell!}\,\frac{(S^\dagger)^k}{k!}\,|j'_a, j'_b, j'_a + j'_b, j'_a + j'_b\rangle = \\ = \binom{2(j'_a + j'_b) + k + 1}{k}^{1/2} \binom{2(j'_a + j'_b)}{\ell}^{1/2} \left|j'_a + \tfrac{k}{2}, j'_b + \tfrac{k}{2}, j'_a + j'_b, j'_a + j'_b - \ell\right\rangle . \tag{29}$$

At the same time, the left-hand side is equal to

$$\frac{(S^\dagger)^k}{k!}\,\frac{1}{\ell!}\,\frac{d^\ell}{d\mu^\ell}\bigg|_{\mu=0} e^{\mu(J_-^a + J_-^b)}\,|j'_a, j'_a\rangle\,|j'_b, j'_b\rangle = \\ = \frac{(a_2^\dagger b_1^\dagger - a_1^\dagger b_2^\dagger)^k}{k!\,\ell!}\,\frac{d^\ell}{d\mu^\ell}\bigg|_{\mu=0} \frac{(a_1^\dagger + \mu a_2^\dagger)^{2j'_a}}{\sqrt{(2j'_a)!}}\,\frac{(b_1^\dagger + \mu b_2^\dagger)^{2j'_b}}{\sqrt{(2j'_b)!}}\,|0\rangle\,, \tag{30}$$

where we have made use of Eq. (17).

In order to find the expansion of a state $|j_a, j_b, j_{tot}, m_{tot}\rangle$ in terms of $|j_a, m_a\rangle\,|j_b, m_b\rangle$ we set

$$k = j_a + j_b - j_{tot} \quad , \quad \ell = j_{tot} - m_{tot} \quad , \quad j'_a = \frac{j_a - j_b + j_{tot}}{2} \quad , \quad j'_b = \frac{-j_a + j_b + j_{tot}}{2}, \tag{31}$$

and equate the right-hand sides of Eqs. (29) and (30):

$$\binom{j_a + j_b + j_{tot} + 1}{j_a + j_b - j_{tot}}^{1/2} \binom{2j_{tot}}{j_{tot} - m_{tot}}^{1/2} |j_a, j_b, j_{tot}, m_{tot}\rangle = \\ = \frac{(a_2^\dagger b_1^\dagger - a_1^\dagger b_2^\dagger)^{j_a + j_b - j_{tot}}}{(j_a + j_b - j_{tot})!(j_{tot} - m_{tot})!}\,\frac{d^{j_{tot} - m_{tot}}}{d\mu^{j_{tot} - m_{tot}}}\bigg|_{\mu=0} \frac{(a_1^\dagger + \mu a_2^\dagger)^{j_a - j_b + j_{tot}}}{\sqrt{(j_a - j_b + j_{tot})!}}\,\frac{(b_1^\dagger + \mu b_2^\dagger)^{-j_a + j_b + j_{tot}}}{\sqrt{(-j_a + j_b + j_{tot})!}}\,|0\rangle\,. \tag{32}$$

The right-hand side is a sum of terms of the form

$$\frac{(a_1^\dagger)^{n_1^a}}{\sqrt{n_1^a!}}\,\frac{(a_2^\dagger)^{n_2^a}}{\sqrt{n_2^a!}}\,\frac{(b_1^\dagger)^{n_1^b}}{\sqrt{n_1^b!}}\,\frac{(b_2^\dagger)^{n_2^b}}{\sqrt{n_2^b!}}\,|0\rangle = \left|j_a = \tfrac{1}{2}(n_1^a + n_2^a), m_a = \tfrac{1}{2}(n_1^a - n_2^a)\right\rangle . \tag{33}$$

A general explicit expression is relatively complicated so we merely illustrate the calculations with a simple example.

### A. Example: $j_a = j_b = \frac{1}{2}$

In the case $j_a = j_b = \frac{1}{2}$ and $j_{tot} = 0$, $m_{tot} = 0$, and Eq. (32) gives:

$$\sqrt{2}\,\left|\tfrac{1}{2}, \tfrac{1}{2}, 0, 0\right\rangle = (a_2^\dagger b_1^\dagger - a_1^\dagger b_2^\dagger)\,|0\rangle = \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle - \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle . \tag{34}$$

In the case $j_a = j_b = \frac{1}{2}$, $j_{tot} = 1$, Eq. (32) simplifies as follows:

$$\binom{2}{1 - m_{tot}}^{1/2} \left|\tfrac{1}{2}, \tfrac{1}{2}, 1, m_{tot}\right\rangle = \frac{1}{(1 - m_{tot})!}\,\frac{d^{1 - m_{tot}}}{d\mu^{1 - m_{tot}}}\bigg|_{\mu=0} (a_1^\dagger + \mu a_2^\dagger)(b_1^\dagger + \mu b_2^\dagger)\,|0\rangle\,. \tag{35}$$

This yields for $m_{tot} = -1, 0, 1$

$$\begin{aligned} \left|\tfrac{1}{2}, \tfrac{1}{2}, 1, -1\right\rangle &= a_2^\dagger b_2^\dagger\,|0\rangle = \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle , \\ \sqrt{2}\,\left|\tfrac{1}{2}, \tfrac{1}{2}, 1, 0\right\rangle &= (a_2^\dagger b_1^\dagger + a_1^\dagger b_2^\dagger)\,|0\rangle = \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle , \\ \left|\tfrac{1}{2}, \tfrac{1}{2}, 1, 1\right\rangle &= a_1^\dagger b_1^\dagger\,|0\rangle = \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle . \end{aligned} \tag{36}$$

## APPENDIX A: REPRESENTATIONS ON VECTORS AND ON OPERATORS

For operators defined in Eq. (3) we now show that, for any $N$-tuple of parameters $\theta_i \in \mathbb{C}$,

$$\hat{a}^\dagger_\alpha \, (e^{\theta_i \mathbb{A}_i})_{\alpha\beta} = e^{\theta_j \hat{A}_j} \, \hat{a}^\dagger_\beta \, e^{-\theta_k \hat{A}_k}, \tag{A1}$$

which can be further multiplied by a vector $(v_\beta)$ from the representation space to find a double-sided action on the corresponding operator $v_\beta \hat{a}_\beta$ (the right-hand side).

To this end, define operator-valued functions

$$\hat{\mathcal{O}}_\beta(\tau) = e^{\tau\theta_j \hat{A}_j} \, \hat{a}^\dagger_\beta \, e^{-\tau\theta_k \hat{A}_k}, \tag{A2}$$

and calculate, with a help of $[\hat{A}_i, \hat{a}^\dagger_\beta] = \hat{a}^\dagger_\alpha (\mathbb{A}_i)_{\alpha\beta}$,

$$\begin{aligned} \frac{d}{d\tau}\hat{\mathcal{O}}_\beta(\tau) &= e^{\tau\theta_j \hat{A}_j} \, [\theta_i \hat{A}_i, \hat{a}^\dagger_\beta] \, e^{-\tau\theta_k \hat{A}_k} \\ &= \theta_i \, e^{\tau\theta_j \hat{A}_j} \, \hat{a}^\dagger_\alpha (\mathbb{A}_i)_{\alpha\beta} \, e^{-\tau\theta_k \hat{A}_k} \\ &= \hat{\mathcal{O}}_\alpha(\tau)(\theta_i \mathbb{A}_i)_{\alpha\beta}. \end{aligned} \tag{A3}$$

Integration of this differential equation, observing the initial condition $\hat{\mathcal{O}}_\beta(0) = \hat{a}^\dagger_\beta$, yields

$$\hat{\mathcal{O}}_\beta(\tau) = \hat{a}^\dagger_\alpha (e^{\tau\theta_i \mathbb{A}_i})_{\alpha\beta}, \tag{A4}$$

therefore proving relation (A1) upon setting $\tau = 1$.

## APPENDIX B: FERMIONIC OPERATORS

Instead of the bosonic operators $a_i, a_i^\dagger$, let us consider $n$ pairs of fermionic operators $f_1, \ldots, f_n$ and $f_1^\dagger, \ldots, f_n^\dagger$ with (canonical) anticommutation relations

$$\{\hat{f}_\alpha, \hat{f}^\dagger_\beta\} = \delta_{\alpha\beta} \quad , \quad \{\hat{f}_\alpha, \hat{f}_\beta\} = 0 \quad , \quad \{\hat{f}^\dagger_\alpha, \hat{f}^\dagger_\beta\} = 0, \tag{B1}$$

and define, for we every $\mathbb{A}_i$, an operator

$$\hat{F}_i = \hat{f}^\dagger_\alpha (\mathbb{A}_i)_{\alpha\beta} \hat{f}_\beta. \tag{B2}$$

Due to the identity

$$[AB, CD] = A\{B, C\}D - AC\{B, D\} + \{A, C\}DB - C\{A, D\}B, \tag{B3}$$

which holds for arbitrary operators $A, B, C, D$, the operators $\hat{F}_i$ form again a representation of the Lie algebra $\mathfrak{g}$:

$$[\hat{F}_i, \hat{F}_j] = c^k_{ij} \hat{F}_k. \tag{B4}$$

Moreover, since

$$[AB, C] = A\{B, C\} - \{A, C\}B, \tag{B5}$$

we have an analogue of Eq. (A1), namely,

$$\hat{f}^\dagger_\alpha \, (e^{\theta_i \mathbb{A}_i})_{\alpha\beta} = e^{\theta_j \hat{F}_j} \, \hat{f}^\dagger_\beta \, e^{-\theta_k \hat{F}_k}. \tag{B6}$$

[1] J. Schwinger, *On Angular Momentum*, Unpublished Report, Number NYO-3071 (1952).
[2] J. Bellissard and R. Holtz, *Composition of coherent spin states*, J Math Phys **15**, 1275 (1974).