Geometric Algebra and Calculus: Unified Language for Mathematics and Physics

Václav Zatloukal*

Faculty of Nuclear Sciences and Physical Engineering,
Czech Technical University in Prague,
Břehová 7, 115 19 Praha 1, Czech Republic

These are notes of an introductory lecture on geometric algebra and calculus presented at L'agape summer school for theoretical physics held 21–28 July 2018 in Mezeyrac, France.

Contents

I. HOW TO MULTIPLY VECTORS – THE GEOMETRIC PRODUCT

We consider a *real* vector space V, and define the *geometric product* ab by the following rules [1, 2] ($a, b, c \in V$):

$$(ab)c = a(bc), \tag{1}$$

$$a(b+c) = ab + ac, \tag{2}$$

$$a^2 = |a|^2, \tag{3}$$

where $|a|$ is a positive scalar called the *magnitude* of a, and $|a| = 0$ implies $a = 0$. [1]

---

*Electronic address: `zatlovac@gmail.com`; URL: `http://www.zatlovac.eu`

[1] Here we presume spaces with positive-definite non-degenerate metric.

The geometric product is not commutative, but we can define the *inner* and *outer* product as, respectively, its symmetric and anti-symmetric part:

$$a \cdot b = \frac{1}{2}(ab + ba), \tag{4}$$

$$a \wedge b = \frac{1}{2}(ab - ba). \tag{5}$$

The inner (or dot) product of two vectors is a scalar, as follows by expanding the scalar quantity $(a+b)^2$. The outer (or wedge) product of two vectors is a new geometric entity called a *bivector*, which represents a directed area spanned by the vectors a and b.

The geometric product can be decomposed into a sum of a scalar and a bivector

$$ab = a \cdot b + a \wedge b. \tag{6}$$

This representation is sometimes used as a definition of the geometric (of Clifford) product, provided the inner and outer products are know a priori. Our construction, however, introduced the geometric product as a single primary concept. [2]

II. GEOMETRIC ALGEBRA OF 2D SPACE

Let $\{e_1, e_2\}$ be an orthonormal basis of a two-dimensional vector space. Geometric product of any pair of vectors can be inferred from the products of basis vectors

$$e_1^2 = e_2^2 = 1 \quad , \quad e_1 e_2 = e_1 \wedge e_2 = -e_2 e_1 \equiv I, \tag{7}$$

with the help of distributivity.

Products with the bivector I are

$$e_1 I = -I e_1 = e_2 \quad , \quad e_2 I = -I e_2 = -e_1 \quad , \quad I^2 = e_1 e_2 e_1 e_2 = -1. \tag{8}$$

The last relation suggests that expressions of the form $\alpha + \beta I$, where α and β are scalars, can be interpreted as complex numbers. (In particular because they always commute, and their set is closed under multiplication.) The important conceptual point here is that the 'unit imaginary' I is not an alien formal entity, but has a clear geometric meaning — it is a unit bivector representing the plane spanned by e_1 and e_2.

The order of basis vectors in I is significant. Eq. (8) tells us that right multiplication by I maps e_1 to e_2, and e_2 to $-e_1$, hence rotating any vector counterclockwise by 90°. In contrast, the unit bivector $e_2 e_1 = -I$ (which also squares to -1) rotates vectors in the opposite direction.

To depict complex numbers $\alpha + \beta I$ as points in the plane we need to choose a unit vector to represent the 'real axis'. For example, taking e_1, and forming

$$u = e_1(\alpha + \beta I) = \alpha e_1 + \beta e_2 \tag{9}$$

renders the real and imaginary part, α and β, as components of the vector u in the $\{e_1, e_2\}$ basis.

Complex numbers are useful in physics whenever 2D rotations take place. In the geometric algebra approach, in order to maintain clear geometric interpretation, we distinguish the object

---

[2] This is also the reason for using the simplest possible symbol (the empty one) to denote the geometric product.

that are being rotated — the vectors, and the object that perform the rotation — the *rotors.* Rotors in 2D are unit complex numbers

$$U_\theta = \cos\theta + I\sin\theta = e^{I\theta}. \tag{10}$$

They naturally act on vectors via right multiplication, e.g.,

$$e_1 U_\theta = e_1\cos\theta + e_2\sin\theta, \tag{11}$$

and analogously for e_2, showing that vectors are, indeed, rotated by angle θ in the counterclockwise direction.

One can visualize a rotor U_θ as a directed arc on a unit circle with length θ, or, more precisely, as an equivalence class of all such arcs, since fixing the starting point would correspond to a specific choice of the real axis.

III. GEOMETRIC ALGEBRA OF 3D SPACE

In 3D there are three orthonormal vectors $\{e_1, e_2, e_3\}$, which can be pairwise multiplied to give rise to three unit bivectors

$$e_1e_2 = e_3I \quad , \quad e_2e_3 = e_1I \quad , \quad e_3e_1 = e_2I, \tag{12}$$

where $I = e_1e_2e_3$ denotes a product of all basis vectors — the *pseudoscalar* of the 3D space.

The pseudoscalar maps vectors to bivectors, and vice versa, via right (or left) multiplication — the so called *duality* operation. For example, to a vector $b = \beta_1e_1 + \beta_2e_2 + \beta_3e_3$ there corresponds a bivector

$$B = bI = \beta_1e_2e_3 + \beta_2e_3e_1 + \beta_3e_1e_2. \tag{13}$$

Duality can be used to express the conventional vector cross product in terms of the wedge product:

$$a \times b = b \wedge a\, I. \tag{14}$$

(Here we use the convention that the inner and outer product have priority before the geometric product.) Only in 3D, however, an antisymmetric product of two vectors can be identified with a vector (via duality). In generic dimension, the antisymmetric wedge product yields a bivector $a \wedge b$.

Rotations are usually presented as taking place *around* a particular *axis.* This is a peculiarity of the three-dimensional place, which does not generalize to higher dimensions. What does generalize is the concept of rotation *in* a particular *plane.* [3]

Suppose, for example, that we wish to rotate a generic 3D vector a in the plane spanned by e_1 and e_2 by an angle θ in the direction 'from e_1 to e_2' to obtain a new vector a'. Geometrically this means that we make a decomposition $a = a_{||} + a_\perp$, where $a_{||}$ lies in the e_1, e_2 plane and $a_\perp$ is perpendicular to it, rotate $a_{||}$, and finally add $a_\perp$, which has been left unchanged. Algebraically,

$$a' = a_{||}e^{\theta e_1e_2} + a_\perp, \tag{15}$$

[3] Of course, in 3D, lines are related to planes via duality.

where e_1e_2 now plays the role of the bivector I in Section II. The right-hand side can be cast in terms of a once we assert the commutativity properties $a_{||}e_1e_2 = -e_1e_2a_{||}$, $a_\perp e_1e_2 = e_1e_2a_\perp$, with the result

$$a' = e^{-\frac{\theta}{2}e_1e_2}a_{||}e^{\frac{\theta}{2}e_1e_2} + e^{-\frac{\theta}{2}e_1e_2}a_\perp e^{\frac{\theta}{2}e_1e_2} = e^{-\frac{\theta}{2}e_1e_2}\, a\, e^{\frac{\theta}{2}e_1e_2}. \tag{16}$$

It is now clear that any bivector defines a rotation $a' = e^{-B/2}\, a\, e^{B/2}$, which takes place in the B-plane, in the sense characterizes by the orientation of B, and by the angle equal to the area (the magnitude) of B. In turn, any rotation can be characterized by a certain bivector. [4]

All unit bivectors square to -1. Hence, if we make the identification

$$i \equiv -e_1e_2 \quad , \quad j \equiv -e_2e_3 \quad , \quad k \equiv -e_3e_1, \tag{17}$$

we reproduce the defining relations of the algebra of quaternions

$$i^2 = j^2 = k^2 = ijk = -1. \tag{18}$$

A generic quaternion is a sum of a scalar and a bivector, and the set of these is closed under multiplication. But, in fact, we don't have to bother about quaternions anymore — all their computational power is encoded in the geometric algebra of the three-dimensional space.

IV. GENERIC MULTIVECTORS

Let us summarize that a two-dimensional vector space V_2 endowed with the geometric product accommodates four linearly independent elements: one scalar 1, two vectors e_1 and e_2, and one bivector e_1e_2. Their linear combinations span the four-dimensional geometric algebra $\mathcal{G}(V_2)$.

For a 3D space V_3, the geometric algebra $\mathcal{G}(V_3)$ consists of eight basis elements: one scalar 1, three vectors e_1, e_2, e_3, three bivectors e_1e_2, e_2e_3, e_3e_1, and one trivector $e_1e_2e_3$.

For vector spaces of generic dimension n the geometric algebra $\mathcal{G}(V_n)$ consists of elements

$$1 \quad , \quad e_i \quad , \quad e_ie_j\ (i<j) \quad , \quad \dots \quad , \quad e_1\dots e_n, \tag{19}$$

where the basis $\{e_i\}_{i=1}^n$ is, for simplicity, chosen orthogonal so that $e_ie_j = e_i \wedge e_j$. The elements are grouped according to the number of vectors being multiplied. There are $\binom{n}{r}$ linearly independent r-vectors that are built of r-tuples of the basis vectors. The total dimension of $\mathcal{G}(V_n)$ is 2^n.

Linear combinations of elements (19) are called *multivectors*. Generic multivector is a sum

$$A = \sum_{r=0}^{n} \langle A\rangle_r, \tag{20}$$

where $\langle A\rangle_r$ are r-vectors, or, *homogeneous* multivectors of *grade* r.

Outer product of r vectors is defined in terms of the geometric product as the totally anitsymmetrized expression

$$a_1 \wedge \dots \wedge a_r = \sum_{\pi\in\mathcal{S}_n} sgn(\pi) a_{\pi(1)}\dots a_{\pi(n)}, \tag{21}$$

---

[4] The two-side representation of rotations, Eq. (16), is a significant achievement of the geometric algebra formalism. We shall have more to say about rotations in arbitrary-dimensional spaces in Section V.

generalizing the definition (5). It vanishes whenever one of the vectors is a linear combination of the remaining vectors (in particular, when two of the vectors coincide). If a_j's are linearly independent, their outer product represents an r-dimensional oriented volume. One can then always find, via the Gram-Schmidt process [1, Ch. 1-3], an orthogonal set of vectors $\hat{a}_j$ to write $a_1 \wedge \ldots \wedge a_r = \hat{a}_1 \ldots \hat{a}_r$. r-vectors of the form (21) are called *blades*. [5]

The inner and outer product can be generalized as follows (a is a vector and A_r an r-vector):

$$\begin{aligned} a \cdot A_r &= \frac{1}{2}(aA_r - (-1)^r A_r a) = (-1)^{r-1} A_r \cdot a, \\ a \wedge A_r &= \frac{1}{2}(aA_r + (-1)^r A_r a) = (-1)^r A_r \wedge a. \end{aligned} \tag{22}$$

The inner product with vector always lowers the grade of A_r by one, while the outer product raises the grade by one (see [2, Ch. 4.1.2] for details). In general, the geometric product of A_r and B_s can be decomposed as

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \ldots \langle A_r B_s \rangle_{r+s}, \tag{23}$$

and the symbols $\cdot$ and $\wedge$ are retained for the lowest and highest grade terms in the series:

$$\begin{aligned} A_r \cdot B_s &= \langle A_r B_s \rangle_{|r-s|}, \\ A_r \wedge B_s &= \langle A_r B_s \rangle_{r+s}. \end{aligned} \tag{24}$$

The wedge product is associative.

The *magnitude* of an r-vector A_r is defined

$$|A_r| = \sqrt{\widetilde{A}_r \cdot A_r}, \tag{25}$$

where $\widetilde{\ }$ is the *reversion* operation defined by $\widetilde{AB} = \widetilde{B}\widetilde{A}$, linearity, and $\widetilde{a} = a$ for vectors. [6] Note that $\widetilde{A}_r = (-1)^{r(r-1)/2} A_r$. The magnitude of a blade $A_r = a_1 \wedge \ldots \wedge a_r$ has direct geometric interpretation. It is the volume of the parallelogram spanned by the vectors $a_1, \ldots, a_r$ (as follows from Formula (28) below).

The inverse of a blade A_r is

$$A_r^{-1} = \frac{\widetilde{A}_r}{|A_r|^2}, \tag{26}$$

since $\widetilde{A}_r \cdot A_r = \widetilde{A}_r A_r$. Note, however, that multivectors in general are not invertible.

Any r-vector A_r defines a scalar-valued r-linear totally antisymmetric linear map (an r-form)

$$\alpha_r(b_1, \ldots, b_r) = \widetilde{A}_r \cdot (b_1 \wedge \ldots \wedge b_r). \tag{27}$$

In particular, for a blade $A_r = a_1 \wedge \ldots \wedge a_r$ one finds

$$\alpha_r(b_1, \ldots, b_r) = \det(a_i \cdot b_j), \tag{28}$$

---

[5] Note, however, that not all homogeneous multivectors are blades. The simplest counter-example is found in dimension 4, where the bivector $\alpha e_1 e_2 + \beta e_3 e_4$ cannot be written as a wedge product of two vectors.

[6] In the formalism, we obey a useful notation convention, which states that scalars are denoted by Greek letters $\alpha, \beta, \ldots$, vectors by lower case Latin letters $a, b, \ldots$, and other multivectors by upper case Latin letters $A, B, \ldots$. However, this convention often clashes with traditional symbols of quantities used in applications.

which is the determinant of the $r \times r$ matrix with entries $a_i \cdot b_j$.

Many algebraic identities can be derived to manipulate multivectors, and hence turn the geometric algebra into an efficient computational language. Since the purpose of these notes is to give a general overview rather than technical exposition, the formulas will not be presented here, and the reader is relegated to Refs. [1, Ch. 1-1] and [2, Ch. 4.1]. Nevertheless, most of the time the identities can be understood, at least qualitatively, on the basis of geometric intuition.

Vectors in geometric algebra can be represented by matrices (cf. Dirac gamma matrices) and the geometric product by matrix multiplication. Although matrices can be used to make contact with traditional treatment of Clifford algebras, they neither clarify the theory nor simplify calculations, and so we don't introduce them at any point.

V. PROJECTIONS, REFLECTIONS, ROTATIONS

Every nonzero r-blade A_r determines a unique r-dimensional linear subspace $\mathcal{A}_r$ of vector space V_n consisting of vectors that satisfy $a \wedge A_r = 0$. Indeed, suppose $\{a_1, \ldots, a_r\}$ is a basis of $\mathcal{A}_r$. Then $a \in \mathcal{A}_r$ if and only if a can be expressed as a linear combination $a = \sum_{i=1}^{r} \alpha_i a_i$, and this in turn is equivalent with the condition $a \wedge A_r = 0$ for $A_r = a_1 \wedge \ldots \wedge a_r$. [7]

The identification of a linear subspace $\mathcal{A}_r$ with a blade A_r can be used to derive an elegant formula for the projection of a vector b onto $\mathcal{A}_r$. Write

$$b = bA_rA_r^{-1} = b \cdot A_rA_r^{-1} + b \wedge A_rA_r^{-1} = (b \cdot A_r) \cdot A_r^{-1} + (b \wedge A_r) \cdot A_r^{-1}. \tag{29}$$

The first term is a vector that lies in $\mathcal{A}_r$, while the second is perpendicular to $\mathcal{A}_r$. [8] Hence, we find the orthogonal *projection onto* $\mathcal{A}_r$

$$b_{||} = b \cdot A_rA_r^{-1}, \tag{30}$$

and *rejection from* $\mathcal{A}_r$

$$b_{\perp} = b \wedge A_rA_r^{-1}. \tag{31}$$

Reflections also enjoy an elegant representation when we employ the geometric product. Take a unit vector u, and reflect a vector a in the plane perpendicular to u:

$$a' = a - 2\, a \cdot u\, u = a - auu - uau = -uau. \tag{32}$$

Reflections are orthogonal transformations, as can be verified by writing

$$a' \cdot b' = \frac{1}{2}(a'b' + b'a') = \frac{1}{2}(uabu + ubau) = u\, a \cdot b\, u = a \cdot b, \tag{33}$$

and have determinant -1.

Composition of reflections along multiple directions $u_1, \ldots, u_r$ is simply

$$a' = (-1)^r U a \widetilde{U} \quad , \quad U = u_r \ldots u_1, \tag{34}$$

---

[7] A given subspace $\mathcal{A}_r$ determines the corresponding blade A_r only up to scalar multiplication, and the blade itself can be presented as an outer product of many different ordered sets of vectors.

[8] The latter claim follows from

$$\big((b \wedge A_r) \cdot A_r^{-1}\big) \cdot a_i = (b \wedge A_r) \cdot (A_r^{-1} \wedge a_i) = 0.$$

is a multivector (called the *versor*) that has the property $U\widetilde{U} = 1$. In turn, every orthogonal transformation can be described as a composition of at most n reflections (the Cartan-Dieudonné theorem).

Composition of an even number of reflections yields an orthogonal transformation with determinant +1 — a *rotation* $a' = Ra\widetilde{R}$, where R is commonly used to denote even versors (*rotors*).

Simple rotation is determined by a pair of unit vectors, $R = vu$. Let $\frac{\theta}{2}$ denote the angle between u and v, and cast

$$R = vu = v \cdot u + v \wedge u = \cos\frac{\theta}{2} - \frac{u \wedge v}{|u \wedge v|}\sin\frac{\theta}{2} = e^{-B/2} \quad , \quad B = \theta\frac{u \wedge v}{|u \wedge v|}. \tag{35}$$

Analogously as in Section III, we find that the rotor (35) realizes a rotation in the plane of bivector blade B through angle $\theta = |B|$ in the direction dictated by the orientation of B. Note that as θ increases by 2π, the rotor R picks up the factor -1. This minus sign drops out once the expression $Ra\widetilde{R}$ is formed.

Generic rotation is a composition of simple rotations. For example,

$$a' = R_2 R_1 a \widetilde{R}_1 \widetilde{R}_2 = Ra\widetilde{R} \quad , \quad R = R_2 R_1, \tag{36}$$

where the total rotor R can be cast, with a help of the Baker-Campbell-Hausdorff formula, as

$$R = R_2 R_1 = e^{-B_2/2} e^{-B_1/2} = e^{-\frac{1}{2}(B_1 + B_2) - \frac{1}{8}(B_1 B_2 - B_2 B_1) + \ldots}, \tag{37}$$

where "…" gathers higher commutators of bivectors B_1 and B_2.

In passing it is convenient to introduce the *commutator product* of multivectors A and B,

$$A \times B = \frac{1}{2}(AB - BA), \tag{38}$$

[9] which satisfies the Jacobi identity

$$A \times (B \times C) + C \times (A \times B) + B \times (C \times A) = 0, \tag{39}$$

and the "Leibniz rule"

$$A \times (BC) = (A \times B)C + B(A \times C). \tag{40}$$

In addition, if B is a bivector and A_r an r-vector, then $B \times A_r$ is again an r-vector, i.e., the operation of taking the commutator product with a bivector is grade-preserving. In particular, we observe that the exponent on the right-hand side of Eq. (37) is a bivector (bivectors form a Lie algebra under the commutator product), and hence a generic rotation can always be presented in the canonical form

$$a' = e^{-B/2}\, a\, e^{B/2} \tag{41}$$

where B is a (not necessarily simple) bivector.

Bivectors naturally define skew-symmetric linear maps

$$\mathsf{B}(a) = a \cdot B. \tag{42}$$

[9] There is no danger in using the same symbol as for the cross product of vectors in 3D, as the latter can be completely replaced by the outer product via Eq. (14), and so we never use it.

Skew-symmetry is easily verified by writing

$$b \cdot \mathsf{B}(a) = b \cdot (a \cdot B) = (b \wedge a) \cdot B = -(a \wedge b) \cdot B = -a \cdot \mathsf{B}(b). \tag{43}$$

Vice versa, to any skew-symmetric map B there is a bivector B that represents it. Namely, if $\{e_i\}_{i=1}^n$ is an orthonormal basis of V_n, then

$$B = \frac{1}{2} e_i \wedge \mathsf{B}(e_i) \tag{44}$$

(sum over i implied) satisfies

$$a \cdot B = \frac{1}{2}\big(a \cdot e_i \mathsf{B}(e_i) - a \cdot \mathsf{B}(e_i) e_i\big) = \frac{1}{2}\big(\mathsf{B}(a) + \mathsf{B}(a) \cdot e_i e_i\big) = \mathsf{B}(a). \tag{45}$$

Skew-symmetric maps form an algebra under commutation. Hence the commutator of two skew-symmetric maps B_1 and B_2 is representable by a bivector, which we find by calculating

$$\mathsf{B}_1(\mathsf{B}_2(a)) - \mathsf{B}_2(\mathsf{B}_1(a)) = (a \cdot B_2) \cdot B_1 - (a \cdot B_1) \cdot B_2 = a \cdot (B_2 \times B_1) \tag{46}$$

to be the commutator product of the corresponding bivectors B_1 and B_2. (We have used the Jacobi identity (39).)

We have seen several examples of linear maps that can be expressed in terms of multivector operations within the geometric algebra: projection, rejection, reflection, rotation, and skew-symmetric mapping. There was no need to use explicit basis or matrix representation to define these maps once we adopted the idea of (geometric) multiplication of vectors.

It is natural to extend linear maps on vectors to linear maps on generic multivectors. Thus, starting with a linear map A, we define the corresponding *outermorphism* by requiring

$$\mathsf{A}(a_1 \wedge \ldots \wedge a_r) = \mathsf{A}(a_1) \wedge \ldots \wedge \mathsf{A}(a_r), \tag{47}$$

for $r = 1, \ldots, n$, $\mathsf{A}(\alpha) = \alpha$ for scalars, and linearity.

In particular, for rotations, the rotor property $R\widetilde{R} = 1$ implies that

$$(Ra_1\widetilde{R}) \wedge \ldots \wedge (Ra_r\widetilde{R}) = Ra_1 \wedge \ldots \wedge a_r\widetilde{R}, \tag{48}$$

i.e., the two-side prescription works for generic multivectors in the same way as for vectors.

For infinitesimal rotation, $R = e^{-\varepsilon B/2}$, we find up to the first order in ε

$$\begin{aligned} Ra_1 \wedge \ldots \wedge a_r\widetilde{R} &\approx a_1 \wedge \ldots \wedge a_r + \varepsilon(a_1 \wedge \ldots \wedge a_r) \times B, \\ (Ra_1\widetilde{R}) \wedge \ldots \wedge (Ra_r\widetilde{R}) &\approx a_1 \wedge \ldots \wedge a_r + \varepsilon \sum_{i=1}^{r} a_1 \wedge \ldots \wedge a_{i-1} \wedge (a_i \cdot B) \wedge a_{i+1} \wedge \ldots \wedge a_r. \end{aligned} \tag{49}$$

Since the two lines coincide, we have derived the identity

$$(a_1 \wedge \ldots \wedge a_r) \times B = \sum_{i=1}^{r} a_1 \wedge \ldots \wedge a_{i-1} \wedge (a_i \cdot B) \wedge a_{i+1} \wedge \ldots \wedge a_r. \tag{50}$$

VI. DIFFERENTIATION

Let $\mathbb{R}^n$ be a space of points x, and V_n a tangent space, over which a geometric algebra $\mathcal{G}(V_n)$ is constructed. Now we can start to talk about the *geometric calculus*, i.e., differentiation and integration that takes advantage of the rich algebraic structure of geometric algebra. [10]

---

[10] We consider a flat underlying space for simplicity, although theory for curved manifolds exists [1, Chs. 4,5,6,7].

The derivative in direction of vector a of a multivector-valued function $F(x)$ is defined as usual:

$$a \cdot \partial F = \lim_{\varepsilon \to 0} \frac{F(x+\varepsilon a) - F(x)}{\varepsilon}. \tag{51}$$

It is a grade-preserving operation.

Choosing an orthonormal frame $\{e_i\}_{i=1}^n$, we define the *vector derivative* [11]

$$\partial F = e_i \, \partial_i F \quad , \quad \partial_i \equiv e_i \cdot \partial. \tag{52}$$

The operator ∂ (also known as the *Dirac operator*) has algebraic properties of a vector, and so the vector derivative splits into two parts,

$$\partial F = \partial \cdot F + \partial \wedge F. \tag{53}$$

If F is vector-valued, then $\partial \cdot F$ is simply the divergence of the vector field F. In 3D the other term $\partial \wedge F$ is directly related to the curl (or 'rot' operation) via Eq. (14), and in generic dimension it corresponds to the exterior derivative [1, Ch. 6-4]. For a scalar field, $\partial \phi$ is the gradient of ϕ. The square of ∂ is the Laplace operator:

$$\partial^2 F = (e_i \partial_i)(e_j \partial_j) F = (e_i \cdot e_j \partial_i \partial_j + e_i \wedge e_j \partial_i \partial_j) F = \sum_{i=1}^n \partial_i^2 F, \tag{54}$$

where the 'wedge' term vanishes due to (assumed) interchangeability of partial derivatives.

To state the general Leibniz rule for vector derivative,

$$\partial(FG) = (\partial F)G + \grave{\partial} F \grave{G}, \tag{55}$$

we have introduced the 'accent' notation. The reason is that the multivector fields F and G, and ∂ don't commute in general, and we want to avoid using an explicit basis as in Eq. (52).

Suppose now we limit ourselves to 2D and consider a complex (in the sense of 2D geometric algebra) function $F(x) = \alpha(x) + \beta(x) I$, $I = e_1 e_2$. The vector derivative operator $\partial = e_1 \partial_1 + e_2 \partial_2$ acts on F as

$$\partial F = e_1 \partial_1 \alpha + e_2 \partial_2 \alpha + e_1 I \partial_1 \beta + e_2 I \partial_2 \beta = e_1(\partial_1 \alpha - \partial_2 \beta) + e_2(\partial_2 \alpha + \partial_1 \beta). \tag{56}$$

If we demand that $\partial F = 0$, we recover the Cauchy-Riemann conditions for holomorphic functions.

In generic dimension, multivector fields that satisfy condition ∂F are referred to as *monogenic* functions. Similarly as holomorphic function, their values in a given region are determined by an integral over the boundary of that region (as we shall see in Section VII below).

VII. INTEGRATION

Directed integral of a multivector-valued function $F(x)$ over a region $\Omega \subset \mathbb{R}^n$ is defined as the (Riemann) infinite sum

$$\int_{x \in \Omega} dX(x) F(x) \equiv \lim_{n \to \infty} \sum_{i=1}^n \Delta X(x_i) F(x_i), \tag{57}$$

---

[11] We could also use a non-orthonormal $\{a_i\}_{i=1}^n$, but then one has to define the *reciprocal frame* [2, Ch. 4.3] $\{a^i\}_{i=1}^n$ consisting of vector that satisfy $a_i \cdot a^j = \delta_i^j$ to expand $\partial = a^i a_i \cdot \partial$.

where $\Delta X(x_i)$ are n-vectors (pseudoscalars of V_n) defined at points x_i, whose magnitude $|\Delta X(x_i)|$ equals the volume of the i-th cell in the n-th discretization of the region Ω.

Integral over submanifolds Σ of $\mathbb{R}^n$ can be defined analogously — the surface element $dS(x)$ is now an r-vector field (for r-dimensional submanifolds) that slides along Σ, and can change its direction.

The directed integral can be used to define the vector derivative (and hence may be regarded as more fundamental concept). Take a point $x_0 \in \mathbb{R}^n$ and a shrinking sequence of neighbourhoods $\mathcal{R}$ of x_0. Then the vector derivative of F at point x_0 is the limit

$$\partial F = \lim_{|R|\to 0} \frac{1}{R} \oint_{\partial\mathcal{R}} dSF \quad , \quad R = \int_{\mathcal{R}} dX = I \int_{\mathcal{R}} |dX| = |R| I, \tag{58}$$

where I is the unit pseudoscalar of V_n. If $\mathcal{R}$ are chosen to be n-dimensional cubes, and $u = I^{-1}dS/|dS|$ denotes the unit normal vector, we easily verify

$$\lim_{|R|\to 0} \frac{1}{R} \oint_{\partial\mathcal{R}} dSF = \lim_{|R|\to 0} \frac{1}{|R|} \oint_{\partial\mathcal{R}} |dS| uF = e_i \partial_i F = \partial F. \tag{59}$$

Note that by setting $F \to I^{-1}GIF$ in Eq. (57), we make sense of the integral $\int_\Omega G dX F$, where F and G are arbitrary multivector-valued functions.

A key achievement of the integration theory is the Fundamental theorem of (geometric) calculus, which states that

$$\int_\Omega \grave{G} dX \grave{\partial} \grave{F} = \oint_{\partial\Omega} G dS F, \tag{60}$$

where Ω is an open region in $\mathbb{R}^n$, and dX and dS are directed volume elements on Ω and $\partial\Omega$, respectively. [12] To show this for the case $G \equiv 1$, one uses the integral representation of the vector derivative, Eq. (58), the definition of directed integral, Eq. (57), and the resolution $\Omega = \bigcup_{i=1}^n \Omega_i$. One finds

$$\int_\Omega dX \partial F = \lim_{n\to\infty} \sum_{i=1}^n \Delta X(x_i) \partial F(x_i) = \lim_{n\to\infty} \sum_{i=1}^n \oint_{\partial\Omega_i} dSF = \oint_{\partial\Omega} dSF. \tag{61}$$

To see the power and generality of the Fundamental theorem, let us consider its three special cases. First, we set $G = I^{-1}$, and consider vector-valued F to obtain

$$\int_\Omega I^{-1} dX \partial F = \int_\Omega \partial F |dX| = \oint_{\partial\Omega} I^{-1} dSF = \oint_{\partial\Omega} uF|dS| \quad , \quad u = I^{-1}dS/|dS|, \tag{62}$$

whose scalar part is the Gauss theorem of vector analysis.

Second, for Ω an m-dimensional embedded manifold choose $G \equiv 1$, consider $m-1$-vector valued F, and restrict to the scalar part to find

$$\int_\Omega (dX \cdot \partial) \cdot F = \int_\Omega dX \cdot (\partial \wedge F) = \int_{\partial\Omega} dS \cdot F, \tag{63}$$

[12] The Fundamental theorem can be generalized from flat $\mathbb{R}^n$ to manifolds [1, Ch. 7-3]. In particular, if Ω is an m-dimensional manifold embedded in $\mathbb{R}^n$ then the only modification of Eq. (60) consists in replacing $dX\dot{\partial}$ by $dX \cdot \dot{\partial}$, which projects the vector derivative of the ambient space onto Ω [2, Ch. 6.4].

which is the celebrated Stokes theorem. [13]

Finally, assume $G(x,x')$ is a vector-valued function of two points (the *Green function* of the vector derivative operator) that satisfies

$$\partial_x G = -\grave{G}\grave{\partial}_{x'} = \delta^{(n)}(x-x'). \tag{64}$$

A use of the Leibniz rule in Eq. (60), and a straightforward rearrangement yield (for $x' \in \Omega$)

$$(-1)^n I F(x') = \int_{x\in\partial\Omega} G(x,x')dS(x)F(x) - \int_{x\in\Omega} G(x,x')dX(x)\partial F(x). \tag{65}$$

If, in addition, the function F is monogenic (i.e., $\partial F = 0$), and we use an explicit expression for the Green function,

$$G(x,x') = \frac{\Gamma(n/2)}{2\pi^{n/2}}\frac{x-x'}{|x-x'|^n}, \tag{66}$$

we arrive at the formula

$$F(x') = (-1)^n \frac{\Gamma(n/2)}{2\pi^{n/2}} I^{-1} \oint_{x\in\partial\Omega} \frac{x-x'}{|x-x'|^n} dS(x)F(x), \tag{67}$$

which expresses the value of a monogenic function F at a point inside the region Ω by an integral over the boundary $\partial\Omega$.

Eq. (67) generalizes the Cauchy integral formula of complex analysis, to which it reduces in two dimensions. Indeed, setting $n = 2$, $F(x) = \alpha(x) + \beta(x)I$, and $z = e_1 x$ (as in Eq. (9)), we find

$$F(x') = \frac{1}{2\pi I}\oint_{x\in\partial\Omega}(x-x')^{-1}e_1 e_1 dS(x)F(x) = \frac{1}{2\pi I}\oint_{\partial\Omega}\frac{F(e_1 z)}{z-z'}dz. \tag{68}$$

VIII. APPLICATIONS

Many physical applications of the above formalism can be found in Ref. [2]. Let us mention, for example, rigid-body motion in Ch. 3.4 (see also [3]), electrodynamics and spacetime physics in Ch. 5 and 7 (see also [4]), differential geometry of embedded manifolds in Ch. 6.5 (see also [1, Ch. 5] or brief summaries [5, 6]), and real Pauli-Schrödinger quantum mechanics in Ch. 8.1 (see also [7]).

[1] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus*, Springer (1987).
[2] C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, Cambridge Univ. Press (2007).
[3] D. Hestenes, *New Foundations for Classical Mechanics*, 2nd Ed., Springer (1999).
[4] D. Hestenes, *Space-Time Algebra*, Gordon and Breach (1966).
[5] D. Hestenes, *The Shape of Differential Geometry in Geometric Calculus*, In: L. Dorst and J. Lasenby (eds.), *Guide to Geometric Algebra in Practice*, Springer, London (2011).
[6] V. Zatloukal, *Shape tensor and geometry of embedded manifolds*, arXiv:1808.03060 (2018).
[7] D. Hestenes, *Oersted Medal Lecture 2002: Reforming the mathematical language of physics*, Am. J. Phys. **71**, 104 (2003).

[13] Of course, to recover the traditional differential-geometric statement of Stokes theorem one has to make connection between multivectors and differential forms, and the curl and exterior derivative [1, Ch. 6-4].