Transport properties of anyons in random topological environment

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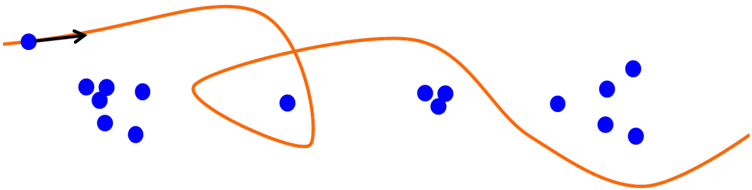
Outline

How does the exchange statistics influence transport properties of anyons?

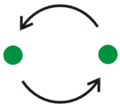


- Anderson localization and anyons
- Definition of the anyonic quantum walk model with randomness
- Abelian anyons and multiple scattering of waves
- Non-Abelian (Ising model) anyons
- Summary of results:
  localization of Abelian, propagation of non-Abelian anyons

V. Zatloukal, L. Lehman, S. Singh, J.K. Pachos, and G.K. Brennen, *Transport properties of anyons in random topological environments*, PRB **90**, 134201 (2014)

Anyons

Indistinguishable particles exchanged

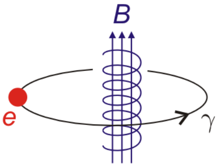


In 3 (or more) dimensions

bosons: 1
fermions: -1

In two spatial dimensions a richer statistical behavior possible.

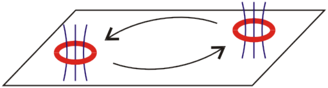
Aharonov-Bohm effect



phase acquired: $e^{iq\Phi}$

$\Phi = \iint_S \mathbf{B} \cdot d\mathbf{S} = \oint_\gamma \mathbf{A} \cdot d\mathbf{l}$

flux-charge composite particles

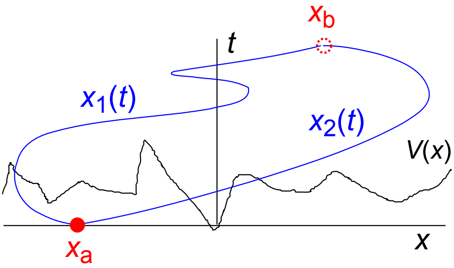


In 2 dimensions

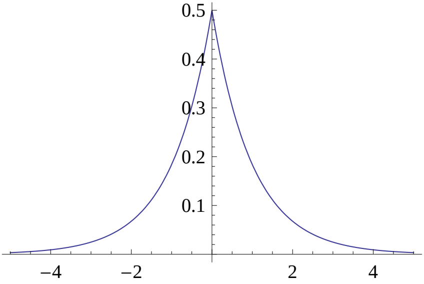
Abelian anyons: phase $e^{i\theta}$
non-Abelian anyons: unitary U

Quasiparticles in *fract. quantum Hall effect*.

Anderson localization



$$\int_{x_a}^{x_b} \mathcal{D}x e^{\frac{i}{\hbar}S[x(t)]} = e^{\frac{i}{\hbar}S[x_1(t)]} + e^{\frac{i}{\hbar}S[x_2(t)]} + \dots$$



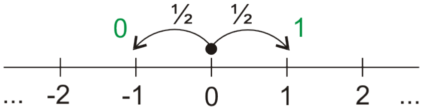
$$|\psi(x, t \to \infty)|^2 \sim e^{-\frac{|x|}{\xi_{loc}}}$$

Anderson localization: particle stops propagating due to destructive interference effect of adding random phases.

$\to$ What about anyonic statistical phases?

P. W. Anderson, Phys. Rev. **109**, 1492 (1958).

Classical random walk (drunken sailor)



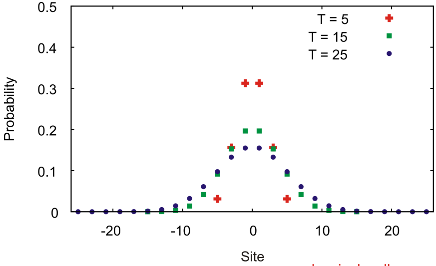
$\rightarrow$ history $\vec{a} = (1, 0, 0, 1, 1, ..., 0)$

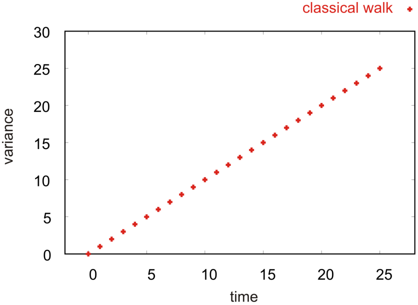
position distribution after t steps

$$P_t(s) = \frac{1}{2^t} \sum_{\vec{a} \to s} 1 = \frac{1}{2^t} \binom{t}{\frac{t+s}{2}}$$

variance

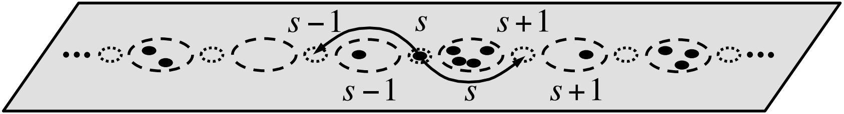
$$\sigma^2(t) = < s^2 > - < s >^2 = t$$





Definition of the anyonic quantum walk model

Quantum analogue of the classical drunken sailor problem.



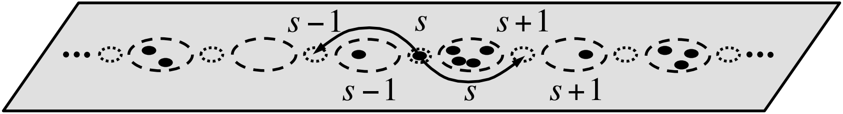
Anyon dynamics:

- Walking anyon moves on a discrete line (coordinate s) in discrete time steps (t)
- Braiding counterclockwise around islands
- Islands are populated by static anyons of the same type as the walker
- Randomness in the (quenched) island occupation numbers m_s

Purely statistical interactions!

Definition of the anyonic quantum walk model

$\leftarrow |0\rangle_{coin}$ $\quad |1\rangle_{coin} \rightarrow$



b_{s-1} $\quad b_s$ $\quad b_{s+1}$ ... braid generators

Hilbert space of the system: $\mathcal{H} = \mathcal{H}_{space} \otimes \mathcal{H}_{coin} \otimes \mathcal{H}_{fusion}$
$\{walker's\ position\} \otimes \{direction\ of\ movement\} \otimes \{statistical\ interactions\}$

Initial localized state: $|\Psi_0\rangle = |s_0\rangle_{space} |c_0\rangle_{coin} |\Phi_0\rangle_{fusion}$

One-step operator: $W = T\,U$, $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ ... Hadamard coin toss

$$T = \sum_s |s-1\rangle\langle s| \otimes |0\rangle\langle 0| \otimes b_{s-1} + |s+1\rangle\langle s| \otimes |1\rangle\langle 1| \otimes b_s$$

Spatial distribution of the walker

- State after t steps: $|\Psi(t)\rangle = W^t|\Psi_0\rangle$
- Reduced spatial density matrix: $\rho_{space}(t) = tr_{coin} tr_{fusion} |\Psi(t)\rangle\langle\Psi(t)|$
- Walker's spatial distribution: $p(s,t) = \langle s|\rho_{space}(t)|s\rangle$

$$p(s,t) = \frac{1}{2^t} \sum_{(\vec{a},\vec{a}')\rightsquigarrow s} (-1)^{z(\vec{a})+z(\vec{a}')} \, tr\mathcal{Y}_{\vec{a}\vec{a}'}$$

$\vec{a} \in \{0,1\}^t \ldots$ a bitstring of walker's moves, e.g. $\vec{a} = (0,1,1,\ldots,0,1)$,

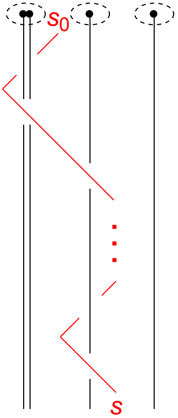
$(\vec{a},\vec{a}') \rightsquigarrow s \ldots$ pairs of paths with $a_t = a'_t$ that lead to site s,

$z(\vec{a}) = \sum_{j=1}^{t-1} a_j a_{j+1}$ for the Hadamard coin and $c_0 = 0$,

$\mathcal{Y}_{\vec{a}\vec{a}'} = B_{\vec{a}}|\Phi_0\rangle\langle\Phi_0|B^\dagger_{\vec{a}'}$,

$B_{\vec{a}}$ is a braidword composed of the braid generators b_s (they depend on m_s), e.g. $B_{\vec{a}} = b_{s_0-1} b_{s_0-1} b_{s_0} \ldots b_{s-1} b_{s-1}$.

b_s's capture (Abelian or non-Abelian) statistics of the particles.



- $b_s = \left(e^{i\phi}\right)^{m_s}$, ϕ is the anyonic exchange angle
- $\mathcal{H}_{fusion} = \mathbb{C}$ is trivial

Variance as a measure of spreading: $\sigma^2(t) = \sum_s p(s,t)s^2 - \left[\sum_s p(s,t)s\right]^2$

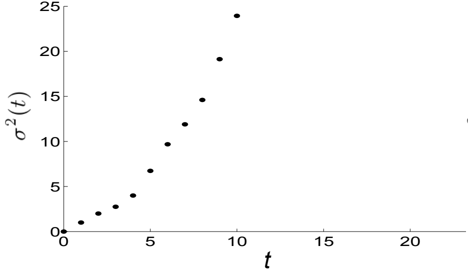
$\langle\langle\sigma^2(t)\rangle\rangle$ ... averaged over occupation numbers m_s

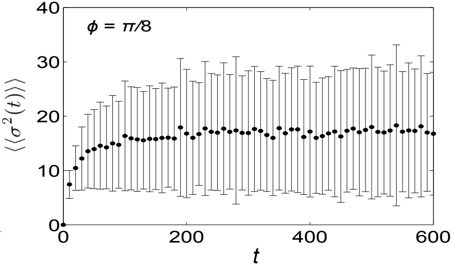
Uniform filling ($m_s = m, \forall s$)

$\sigma^2(t) \sim t^2$ ... ballistic propagation

Random filling (m_s's are random)

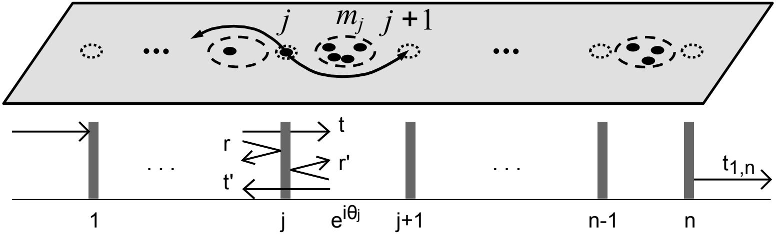
$\langle\langle\sigma^2(t)\rangle\rangle \sim const$ ... localization





Randomization of phases → destructive interference → **localization**

Abelian quantum walk as multiple scattering of waves



Abelian anyonic quantum walk

- Step of the walk:
  coin operator $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\leftrightarrow$
- Randomness in island occupation numbers m_j $\leftrightarrow$
- Statistical phase upon braiding around an island j: ϕm_j $\leftrightarrow$
- Position distr. $p(n, t \to \infty)$ $\leftrightarrow$

Multiple scattering of waves

- Scattering event:
  scattering matrix $\begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$
- Randomness in distances between scatterers
- Phase acquired upon traveling from scatterer j to $j+1$: θ_j
- Overall transmission prob. $|t_{1,n}|^2$

Abelian quantum walk as multiple scattering of waves

C.A. Müller and D. Delande, *Disorder and interference: localization phenomena*, arXiv:1005.0915v2

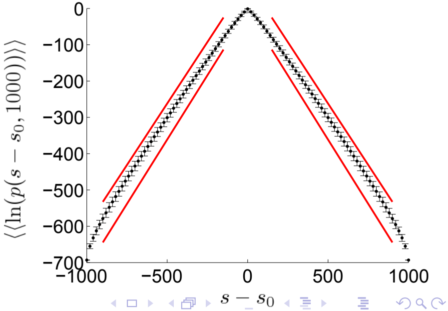
- Assume $\phi = \frac{\pi}{N}$ $(N > 2)$, $\theta_j \in \{0, \frac{\pi}{N}, \ldots, (N-1)\frac{\pi}{N}\}$ uniformly distributed
- Transmission amplitude through scatterers $1, \ldots, n$ expressed recursively:
  $$t_{1,n} = t_{1,n-1} e^{i\theta_{n-1}} \left(1 + r_n e^{i\theta_{n-1}} r'_{1,n-1} e^{i\theta_{n-1}} + \ldots\right) t_n = \frac{t_{1,n-1} e^{i\theta_{n-1}} t_n}{1 - r_n r'_{1,n-1} e^{i2\theta_{n-1}}}$$
- Statistical averaging:
  $\langle\langle \ln |t_{1,n}|^2 \rangle\rangle = \langle\langle \ln |t_{1,n-1}|^2 \rangle\rangle + \ln |t_n|^2 - \langle\langle \ln |1 - r_n r'_{1,n-1} e^{i2\theta_{n-1}}|^2 \rangle\rangle$
- Bounds on the localization length: $\frac{1}{\ln(1+|r|^N)^{\frac{2}{N}} - \ln |t|^2} \leq \xi_{loc} \leq \frac{1}{\ln(1-|r|^N)^{\frac{2}{N}} - \ln |t|^2}$

Transmission exponentially suppressed

$N = 8$, $\phi = \frac{\pi}{8}$, $\begin{pmatrix} r & t' \\ t & r' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$:

$$\exp\langle\langle \ln |t_{1,n}|^2 \rangle\rangle \sim e^{-\frac{n}{\xi_{loc}}}$$

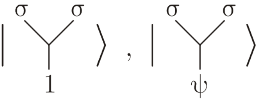
with $1.412 \leq \xi_{loc} \leq 1.477$.



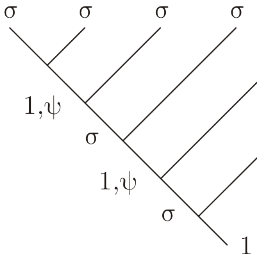
Non-Abelian model: Ising anyons

3 particle types: 1 (vacuum) , σ (anyon) , ψ (fermion)

fusion rules: $\sigma \times \sigma = 1 + \psi$, $\sigma \times \psi = \sigma$, $\psi \times \psi = 1$, $1 \times a = a$



$\in \mathcal{H}^{(2)}_{fusion}$ — *topological Hilbert space*



$\equiv \; | \begin{smallmatrix} 1 \\ \psi \end{smallmatrix} , \begin{smallmatrix} 1 \\ \psi \end{smallmatrix} \rangle \in \mathcal{H}^{(6)}_{fusion}$

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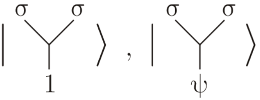
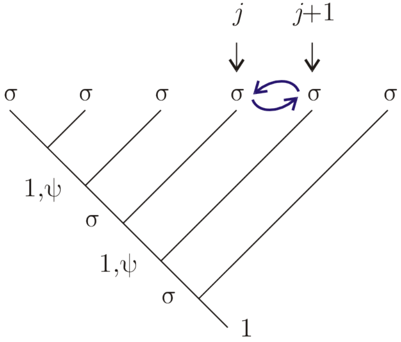


$\in \mathcal{H}^{(2)}_{fusion}$ — *topological Hilbert space*



$\equiv \left| \begin{matrix} 1 \\ \psi \end{matrix} , \begin{matrix} 1 \\ \psi \end{matrix} \right\rangle \in \mathcal{H}^{(6)}_{fusion}$

Non-Abelian model: Ising anyons

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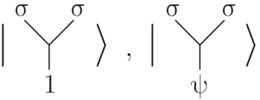
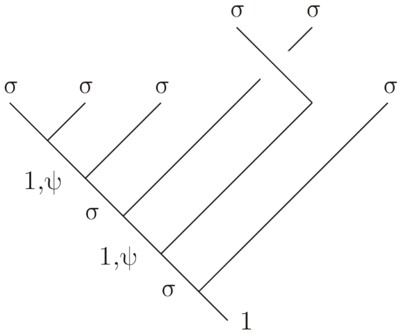


$\in \mathcal{H}^{(2)}_{fusion}$ — *topological Hilbert space*



$\equiv B_j \, | \, {1 \atop \psi} \, , \, {1 \atop \psi} \, \rangle \in \mathcal{H}^{(6)}_{fusion}$

$\uparrow$

unitary representation of the braid group

($\rightarrow$ quantum computation schemes)

Non-Abelian model: Ising anyons

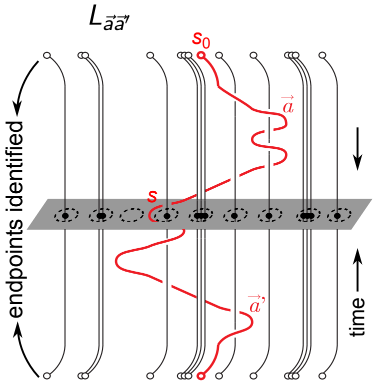
- $\{b_s\}$ are unitary matrices (representation of the braid group)
- dim $\mathcal{H}_{fusion}$ grows exponentially in the number of anyons
- $tr\mathcal{Y}_{\vec{a}\vec{a}'} = \frac{\langle L_{\vec{a}\vec{a}'}\rangle(q^{-1/4})}{d^{|\vec{m}|}}$, $\langle L_{\vec{a}\vec{a}'}\rangle(.)\ldots$ Kauffman bracket of a link $L_{\vec{a}\vec{a}'}$

Ising anyons: $q = i$, $d = \sqrt{2}$

$$\Rightarrow tr\mathcal{Y}_{\vec{a}\vec{a}'} = \prod_{\substack{s=1\\ m_s>0}}^{n} (-i)^{\frac{\ell_s}{2} m_s} \delta_{0,\ell_s \mod 2} \times \prod_{1\leq s'<s''\leq n} (-1)^{m_{s'} m_{s''} \tau(s',s'')}$$

$\ell_s \ldots$ linking numbers between the walker's component and "static" components

$\tau(s', s'') \ldots$ Milnor triple component invariant



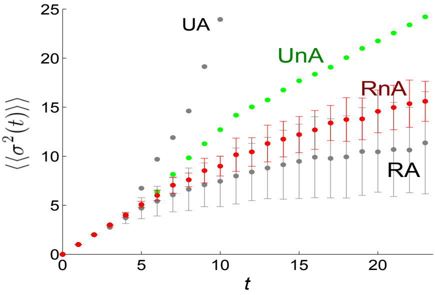
Non-Abelian model: Ising anyons

Uniform filling $m_s = 1, \forall s$

$$tr\mathcal{Y}_{\vec{a}\vec{a}'} = \prod_{s=1}^{n} \delta_{0,\ell_s \mod 2} \times \prod_{1\leq s'<s''\leq n}(-1)^{\tau(s',s'')}$$

$\sigma^2(t) \sim t \ldots$ diffusive propagation

L. Lehman, V. Zatloukal, G. K. Brennen, J. K. Pachos, and Z. Wang, PRL **106**, 230404 (2011)



Random filling

$$\langle\langle tr\mathcal{Y}_{\vec{a}\vec{a}'}\rangle\rangle = \mathcal{T}_{\vec{a}\vec{a}'} \prod_{s=1}^{n} \delta_{0,\ell_s(\vec{a},\vec{a}') \mod 8}$$

$$\mathcal{T}_{\vec{a}\vec{a}'} = \frac{1}{2^n} \sum_{\vec{m}\in\{0,1\}^n} \prod_{1\leq s'<s''\leq n} (-1)^{m_r m_s \tau(s',s'')}$$

$\langle\langle\sigma^2(t)\rangle\rangle \sim t \ldots$ diffusive

Entanglement with fusion degrees of freedom

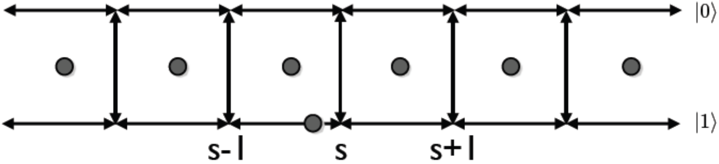
$\rightarrow$ decoherence, interference effects suppressed

$\rightarrow$ **diffusive propagation**

Continuous-time anyonic quantum walk

To allow for longer time simulations,

Hubbard model for anyons on a ladder



Hamiltonian:

$\sum_s [|s+1\rangle\langle s| \otimes |0\rangle\langle 0| \otimes b_s + |s+1\rangle\langle s| \otimes |1\rangle\langle 1| \otimes b_s + h.c.] + |0\rangle\langle 1| + |1\rangle\langle 0|$

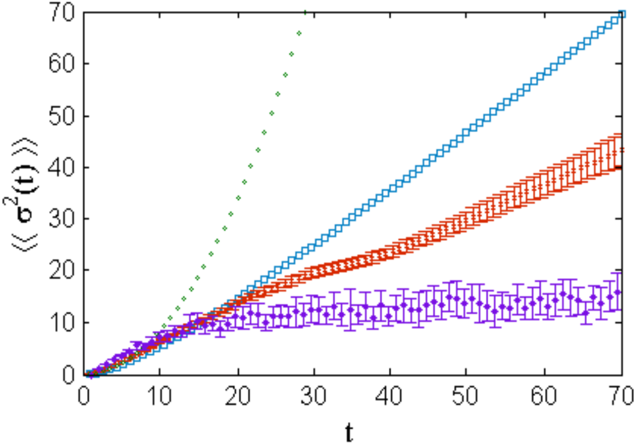
hopping with braiding — coin flip

Numerical studies: real time evolution of anyonic Matrix Product States ("Time-Evolving Block Decimation" algorithm)

S. Singh, R.N.C. Pfeifer, G. Vidal, and G.K. Brennen, PRB **89**, 075112 (2014)

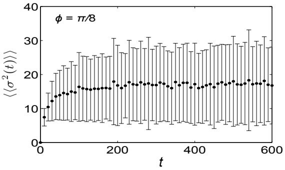
Continuous-time anyonic quantum walk

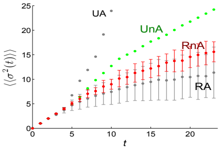
$n = 100$ anyons

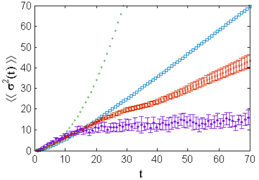


Non-Abelian Ising anyons **do not localize** in the presence of random topological environment!

Summary of results and the moral







	uniform environment	disordered environment
Abelian anyons	ballistic	localizing
Non-Abelian Ising anyons	diffusive	diffusive

By purely statistical interactions, random topological environment localizes Abelian anyons but not non-Abelian anyons.

Thank you for your attention.