Symmetries and gauge fields in the Hamiltonian constraint formulation of classical field theories

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Motivation I: Unification of fundamental interactions

Faraday-Maxwell's field theory of electromagnetism
$\rightsquigarrow$ Einstein's field theory of gravity (general relativity)

(interactions at distance replaced by interactions mediated by fields)
[H. Goenner, *On the History of Unified Field Theories*, Living Rev. Relativity **7**, 2 (2004)]

Nowadays we have:

Electromagnetism $\oplus$ **weak** $\oplus$ **strong** interaction
(Yang-Mills theories with gauge groups $U(1)$, $SU(2)$ and $SU(3)$, resp.)

$\oplus$ **Gravity** (see *Gauge Theory Gravity*)
[A. Lasenby, C. Doran, S. Gull, *Gravity, gauge theories and geometric algebra*, Phil. Trans. Roy. Soc. Lond. A **356** (1998) 487-582, arXiv:gr-qc/0405033]

My aim: Understand the relation between Gravity and Yang-Mills on the level of classical field theory.

Motivation II: Quantization in Hamiltonian formalism

Non-relativistic mechanical system with Hamiltonian $H_0(\mathbf{x}, \mathbf{p})$:
Hamilton-Jacobi $\rightarrow$ [quantization: $\hat{\mathbf{p}} = -i\hbar\,\partial/\partial\mathbf{x}$] $\rightarrow$ **Schrödinger** eq.

$$\frac{\partial S}{\partial t} + H_0(\mathbf{x}, \frac{\partial S}{\partial \mathbf{x}}) = 0 \quad \rightarrow \quad \left[-i\hbar\frac{\partial}{\partial t} + H_0(\mathbf{x}, -i\hbar\frac{\partial}{\partial \mathbf{x}})\right]\psi(\mathbf{x}, t) = 0 \quad (1)$$

Field theoretic Hamiltonian formalism:
[M. J. Gotay et. al, arXiv:physics/9801019v2, arXiv:math-ph/0411032]

1) **canonical**: ∞-dim. space of field configurations $\rightarrow$ functional Schr. eq.

2) **covariant**: finite-dim. configuration space, generalized momentum, classical De Donder-Weyl theory $\rightarrow$ precanonical Schr. eq.
[I. V. Kanatchikov, Rep. Math. Phys. **43** (1999) 157, arXiv:9810165, arXiv:1312.4518]

My talk: **Hamiltonian constraint** formalism ($\sim$ covariant)

Outline

- Unifying spacetime and the space of fields
- Hamiltonian constraint formulation of field theories
  - Canonical equations of motion
  - Local Hamilton-Jacobi theory
  - Symmetries and Hamiltonian Noether theorem
  - Local symmetries and the gauge field
- Example: Scalar field coupled to gravity and Yang-Mills field

[V.Z., Int. J. Geom. Methods Mod. Phys. 13, 1650072 (2016), arXiv:1504.08344]

[V.Z., Adv. Appl. Clifford Algebras 27, 829-851 (2017), arXiv:1602.00468]

[V.Z., arXiv:1611.02906 (2016)]

Mathematical language: Geometric algebra

Geometric algebra and calculus

[D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus*, Springer (1987)]

[C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, Cambridge Univ. P. (2007)]

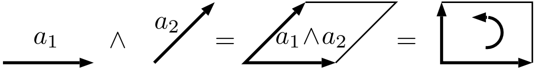
($\Leftrightarrow$ Clifford algebra, Dirac algebra of γ-matrices)

Geometric product: $ab = a \cdot b + a \wedge b$

– associative, invertible, non-commutative

$(\cdot)$ **inner product**: $a \cdot b = b \cdot a$

$(\wedge)$ **outer product** $a \wedge b = -b \wedge a$

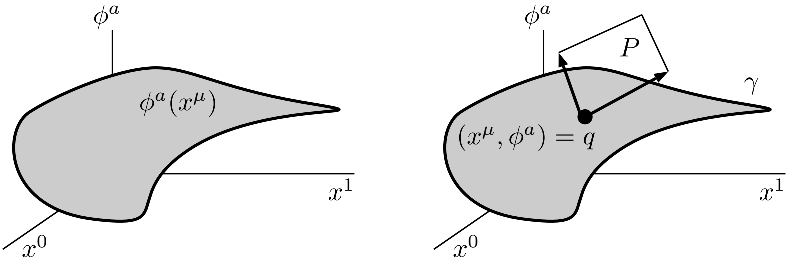


$\rightarrow$ Many useful tools: multivector derivative, rotors, bivector algebras, ...

$\rightarrow$ Coordinate-free differential geometry

Unifying spacetime and the field space

Fields as functions $\Phi(x)$ $\rightarrow$ surfaces $\gamma = \{q = (x, y) \,|\, g(x, y) = 0\}$
$[x = (x^\mu), y = (\phi^a)]$



(cf. special relativity: $t, \mathbf{x} \rightarrow x^\mu \in$ Minkowski space)

$x^\mu, \phi^a \ldots$ *partial observables*
$\mathcal{C} = \{q\} \ldots$ *configuration space* – $N + D$-dimensional
$\gamma \subset \mathcal{C} \ldots$ *motions* – D-dim. surfaces (correlations among partial observ.)
[Chap. 3 in C. Rovelli, *Quantum Gravity*, Cambridge Univ. Press (2004)]

Variational principle with Hamiltonian constraint

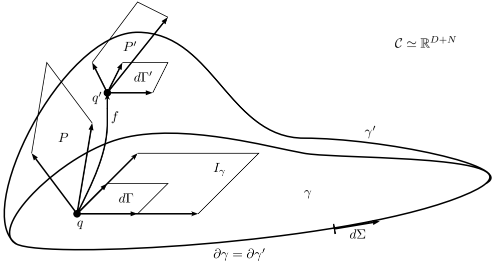
Variational principle

Extremize

$$\mathcal{A}[\gamma, P] = \int_\gamma P(q) \cdot d\Gamma(q) \quad (2)$$

under **Hamiltonian constraint**

$$H(q, P(q)) = 0 \qquad \forall q \in \gamma. \quad (3)$$



$d\Gamma$ ... oriented surface element of γ

P ... multivector of grade D

(H is a generic function of q and P, *not* energy or its density!)

Non-relativistic mechanics ... $H = p \cdot e_t + H_0(q, p_x)$

Scalar field theory ... $H = P \cdot I_x + \frac{1}{2} \sum_{a=1}^{N} \left(I_x \cdot (P \cdot e_a) \right)^2 + V(y)$

String theory ... $H = \frac{1}{2}(|P|^2 - \Lambda^2)$

Classical field theory from Hamiltonian constraint

Lagrange multiplier of Hamiltonian constraint: $\mathcal{A}[\gamma, P, \lambda] = \int_\gamma [P \cdot d\Gamma - \lambda(q) H(q, P)]$

→ **Canonical equations of motion**:

$$\lambda\, \partial_P H(q, P) = d\Gamma \tag{4a}$$

$$(-1)^D \lambda\, \dot{\partial}_q H(\dot{q}, P) = \begin{cases} d\Gamma \cdot \partial_q P & \text{for } D = 1 \text{ (particles)} \\ (d\Gamma \cdot \partial_q) \cdot P & \text{for } D > 1 \text{ (fields)} \end{cases} \tag{4b}$$

$$H(q, P) = 0 \tag{4c}$$

→ **Local Hamilton-Jacobi theory**:

$$H(q, \partial_q \wedge S) = 0 \tag{5}$$

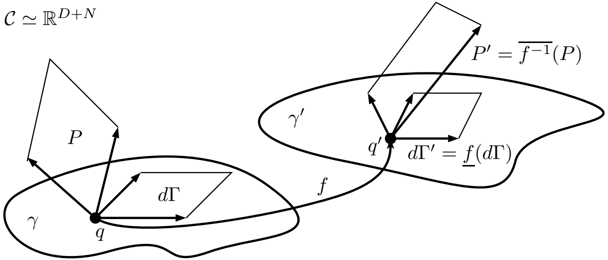
(*Partial* diff. eq., $S(q)$ is not classical action $\mathcal{A}_{cl}[\partial\gamma]$)

[V.Z., Int. J. Geom. Meth. Mod. Phys. **13**, 1650072 (2016), arXiv:1504.08344]

[F. Hélein and J. Kouneiher, J. Math. Phys. **43**, 2306 (2002), arXiv:0010036]

Symmetries and conservation laws

(Active) **Transformation** $q' = f(q)$:



$\underline{f}(a) = a \cdot \partial_q f(q)$ (push-forward), $\overline{f^{-1}}(b) = \partial_q f^{-1}(q) \cdot b$ (pull-back)

Symmetry if: $H(q', P') = H(q, P) \quad (\to \mathcal{A}[\gamma', P', \lambda'] = \mathcal{A}[\gamma, P, \lambda])$

$\to$ **Conservation law**: (corresp. to infsm. sym. $f(q) = q + \varepsilon v(q)$)

$$D > 1: \ (d\Gamma \cdot \partial_q) \cdot (P \cdot v) = 0 \quad , \quad D = 1: \ d\Gamma \cdot \partial_q (P \cdot v) = 0 \qquad (6)$$

$P \cdot v \ldots$ conserved multivector of grade $D - 1$ ($\sim$ Noether current)

[V.Z., Adv. Appl. Cliff. Alg. 27, 829-851 (2017), arXiv:1602.00468, arXiv:1604.03974]

Local transformations

Assume $H(q', P) = H(q, P)$ and define the **Gauged Hamiltonian**

$$H_h(q, P) = H(q, \overline{h}(P; q)) \tag{7}$$

where $\overline{h}$ is the adjoint of a q-dependent linear mapping h (cf. vielbein)

Transformation rule for the (static) **gauge field** h:

$$\overline{h'}(b; q') = \overline{h}\big(\overline{f}(b; q); q\big) \quad (\forall b) \tag{8}$$

Then: $H_{h'}(q', P') = H_h(q, P)$

$\Rightarrow$ classical motions of H_h are transformed to classical motions of $H_{h'}$

[V.Z., arXiv:1611.02906 (2016)]

Gauge field strength

Canonical equations of motion for $H_h(q, P)$: (denote $\bar{P} = \overline{h}(P)$)

$$\lambda\,\partial_{\bar{P}} H(q, \bar{P}) = h^{-1}(d\Gamma) \tag{9a}$$

$$(-1)^D \lambda\,\overline{h}(\dot{\partial}_q) H(\dot{q}, \bar{P}) - h^{-1}(d\Gamma) \cdot \overline{h}\big(\dot{\partial}_q \wedge \dot{\overline{h^{-1}}}(\bar{P})\big) = \begin{cases} d\Gamma \cdot \partial_q \bar{P} & \text{for } D = 1 \\ h^{-1}(d\Gamma \cdot \partial_q) \cdot \bar{P} & \text{for } D > 1 \end{cases} \tag{9b}$$

$$H(q, \bar{P}) = 0 \tag{9c}$$

Field strength: linear mapping from n to $n+1$-vectors

$$F(\bar{P}) = -\overline{h}\big(\dot{\partial}_q \wedge \dot{\overline{h^{-1}}}(\bar{P})\big) \tag{10}$$

Gauge invariance: $F'(\bar{P}') = F(\bar{P})$

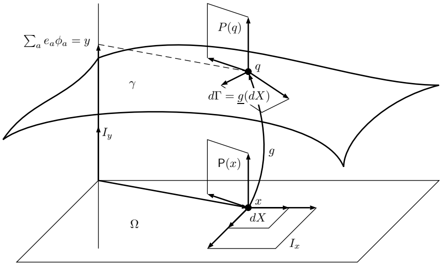
Coordinate frame $\{e_\mu\}_{\mu=1}^{D+N}$: $h_j{}^\mu = \gamma_j \cdot \overline{h}(e^\mu)$, $h^j{}_\mu = \gamma^j \cdot h^{-1}(e_\mu)$

Field strength as **torsion**: $T^\rho{}_{\mu\nu} = h_j{}^\rho \partial_\mu h^j{}_\nu - h_j{}^\rho \partial_\nu h^j{}_\mu = h^{-1}(e_\mu \wedge e_\nu) \cdot F(\overline{h}(e^\rho))$

Example: Scalar field

D-dim. spacetime: $x = (x^\mu)$
N-dim. field space: $y = (\phi_1, \dots, \phi_N)$
Lagrangian formulation:

$$\mathcal{A}_{SF}[y(x)] = \int d^D x \left[\frac{1}{2}(\partial_\mu y)\cdot(\partial^\mu y) - V(y)\right]$$



Hamiltonian constraint for scalar field coupled to a gauge field:

$$H_{SF,h}(q,P) = \overline{h}(P)\cdot I_x + \frac{1}{2}\sum_{a=1}^{N}\left(I_x\cdot(\overline{h}(P)\cdot e_a)\right)^2 + V(y) \qquad (11)$$

Eliminating momentum P (using the first canonical equation (9a))

$$\mathcal{A}_{SF,h}[\gamma] = \int_\gamma \left[\frac{1}{2\lambda}\sum_{a=1}^{N}\left(I_x^{-1}\cdot(h^{-1}(d\Gamma)\cdot e_a)\right)^2 - \lambda V(y)\right], \; \lambda = I_x^{-1}\cdot h^{-1}(d\Gamma)$$

Scalar field and gravity

Spacetime diffeomorphisms: $q' = f_{Gr}(q) = f_x(x) + y$

Gravitational gauge field: $\overline{h}_{Gr}(b;q) = \overline{h}_x(b_x;x) + b_y$

Transformation rule: $\overline{h'_x}(b_x;x') = \overline{h}_x\big(\overline{f}_x(b_x;x);x\big)$

We calculate: $\lambda_{Gr} = |dX|\,\det(h_{Gr}^{-1})$

Scalar field in gravitational background:

$$\mathcal{A}_{SF,h_{Gr}}[y(x)] = \int_\Omega \det(h_{Gr}^{-1}) \left[\frac{1}{2}\sum_{a=1}^{N}\big(\overline{h}_{Gr}(\partial_x\phi_a)\big)^2 - V(y)\right]|dX| \quad (12)$$

Local field-space rotations: $q' = f_{YM}(q) = R(x) q \widetilde{R}(x)$, $R(x) = e^{-B_y(x)/2}$
(Assuming $V(y') = V(y)$)

Yang-Mills gauge field: $\overline{h}_{YM}(b; q) = \widetilde{S}(x) b S(x) - \gamma^\mu \big(y \cdot A_\mu(x)\big) \cdot b$

$\{\gamma^\mu\} \dots$ orthogonal spacetime basis
$\{A_\mu\} \dots$ field-space bivectors ($\Leftrightarrow$ skew-symm. map $\mathsf{A}_\mu : y \mapsto y \cdot A_\mu$)
$S \dots$ field-space rotor

Transformation rules: $S' = RS$, $A'_\mu = RA_\mu \widetilde{R} - 2R\partial_\mu \widetilde{R}$

We calculate: $\lambda_{YM} = |dX|$

Scalar field in Yang-Mills background:

$$\mathcal{A}_{SF,YM}[y(x)] = \int_\Omega \left[\frac{1}{2}(\partial_\mu y + y \cdot A_\mu) \cdot (\partial^\mu y + y \cdot A^\mu) - V(y) \right] |dX| \tag{13}$$

Dynamics of the gauge field

Field strength corresponding to gauge field h: $F(b) = -\overline{h}\big(\dot{\partial}_q \wedge \overline{\dot{h^{-1}}}(b)\big)$

Gravitational field strength (cf. *teleparallel gravity*)

$$F_{Gr}(b) = -\overline{h}_x\big(\dot{\partial}_x \wedge \overline{\dot{h}_x^{-1}}(b_x)\big) \tag{14}$$

Yang-Mills field strength

$$\begin{aligned} F_{YM}(b) = {} & \frac{1}{2}\gamma^\mu \wedge \gamma^\nu \left[y \cdot (\partial_\nu A_\mu - \partial_\mu A_\nu - A_\mu \times A_\nu)\right] \cdot (Sb\widetilde{S}) \\ & + \gamma^\mu \wedge \big((\widetilde{S} A_\mu S - 2\widetilde{S}\partial_\mu S) \cdot b\big) \end{aligned} \tag{15}$$

Q: Find a kinetic term for the unified gauge field h that reduces to

1) Gravitational kinetic term: $\det(h_{Gr}^{-1})\, R$

2) Yang-Mills kinetic term: $Tr(\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu})$

Detour: Geometry of embedded surfaces

Embedded manifold $\mathcal{M}^n \subset \mathbb{R}^N$ with pseudoscalar $I_{\mathcal{M}}(x)$

Shape tensor: $S(a) = I_{\mathcal{M}}^{-1} a \cdot \partial I_{\mathcal{M}}$, a is tangent vector

Covariant derivative: $a \cdot DM = a \cdot \partial M - M \times S(a)$
(Shape $S \to$ Christoffels Γ and Gauge potentials A?)

$\to$ **Curvature**:

$$S(a) \times S(b) = R(a \wedge b) + F(a \wedge b) \tag{16}$$

$R \ldots$ intrinsic, $F \ldots$ external

[D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus*, Springer (1987)]

The last slide

Similarities between Gravity and Yang-Mills (gauge theory structure)
$\rightarrow$ Seek a unified theory of a single gauge field

Unify spacetime and field space $+$ Hamiltonian constraint formulation
$+$ Requirement of invariance under generic transformations
$\rightarrow$ static gauge field h

Q: Find kinetic term for the unified gauge field h.
Q: What about fermions?
Q: What about quantization?

Thank you for your attention.